Infinite Paths in the Situation Calculus: 
Axiomatization and Properties

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1. Motivation
- The situation calculus (McCarthy&Hayes-69)
  - very popular formalism for modeling dynamic domains
  - much has been done to further enrich this language (Reiter-01)
- However, it lacks a convenient way of dealing with "infinite futures"
  - cannot directly talk about such infinite paths in the sit. calc.
  - previous approaches have drawbacks

2. Previous Work
- Introduced a new sort of finite paths modeled by a pair of situations
  - Proved that: $\cup D$ (McCarthy&Hayes-69)
  - Also need a unique-names axiom for paths:
- Formalize paths by making them correspond to action selection
  - proposed an axiomatization of infinite paths
  - have path variables that can be quantified over
  - cannot directly talk about such infinite paths in the sit. calc.
- If $D \Sigma$ – For any executable situation, there is a path that starts with that situation provided
  - $\exists F, s. (\text{OnPath}(F, s) \land \text{OnPath}(p, s) \land p \neq p')$
  - $\forall s, p, s'. (\text{OnPath}(p, s) \land \text{OnPath}(p, s') \land s \preceq s' \land s' \neq s \land s \neq s' \land \text{OnPath}(p, s) \land \text{OnPath}(p, s')) \Rightarrow s_1 \neq s_2$.
  - For any situation $s$ on path $p$, there is a suffix of $p$ that starts with $s$:
  - $\forall p, s. \text{OnPath}(p, s) \land \exists p'. \text{Suffix}(p', p, s)$.
  - Any path that starts with a non-initial situation can be extended in the past:
  - $\forall s, a, s_2. (s_1 = s_2 \land \text{Starts}(p_2, s_2)) \Rightarrow (\text{Starts}(p_1, s_1) \land \text{Suffix}(p_2, p_1, s_2))$.
  - See paper for other properties

3. Contributions
- Introduced a new sort of infinite paths in the situation calculus
  - have path variables that can be quantified over
  - proposed an axiomatization of infinite paths
- Proved that:
  - our paths have some desirable properties
  - proposed axiomatization is correct

- $\text{OnPath}(p, s)$: situation $s$ is on the path $p$
- $\text{Starts}(p, s)$: $s$ is the earliest situation on path $p$
- Formalize paths by making them correspond to action selection function and starting situation pairs: $(F, s)$:
  - $\forall s, F, \text{Executable}(F, s) \land \forall s'. \text{OnPath}(p, s') \Rightarrow (\exists p. \text{OnPath}(F, s) \land \text{OnPath}(p, s))$
  - $\forall F, s. \text{Executable}(F, s) \Rightarrow (\exists s'. \text{OnPath}(p, s') \land \forall s'. \text{OnPath}(F, s') \Rightarrow \text{OnPath}(p, s'))$
  - $\text{OnPath}(F, s, s')$: the situation sequence defined by the strategy $F$ and starting situation $s$ is executable
  - $\forall p, p'. (\forall s. \text{OnPath}(p, s) \equiv \text{OnPath}(p', s) \Rightarrow p = p')$
- $\Sigma \cup D_{\text{path}}$:
- $\forall s, a, s_2. (s_1 = s_2 \land \text{Starts}(p_2, s_2)) \Rightarrow (\text{Starts}(p_1, s_1) \land \text{Suffix}(p_2, p_1, s_2))$.

5. Properties of Infinite Paths in the Sit.Calc. (I)
- Some notations:
  - $\Sigma$: foundational axioms for defining the structure of situations
  - $D_{\text{path}}$: our axiomatization of paths
  - $\text{Suff}(p', p, s)$: $p'$ is suffix of $p$ starting from $s$ if $s$ is on $p$, and $p'$ starts with $s$, which starts with $s$, contains exactly the same situations as $p$ starting from $s$, i.e.:
    - $\forall a, F. \text{OnPath}(p, s) \land \text{Starts}(p', s) \land \forall s'. s \preceq s' \Rightarrow (\forall s. \text{OnPath}(p, s) \equiv \text{OnPath}(p', s'))$
  - $\Sigma \cup D_{\text{path}}$:
    - For any executable situation, there is a path that starts with that situation provided that for any situation there is at least one executable action:
      - $\forall s, a. \text{Poss}(a, s) \Rightarrow (\forall s. \text{Executable}(s) \Rightarrow \exists p. \text{Starts}(p, s))$.
    - The successor situation of a situation on a path is unique:
      - $\forall p, s, a. \text{OnPath}(p, s) \land \text{OnPath}(p, (p, s)) \Rightarrow a = b$.
    - Any pair of situations on the same path are "co-linear":
      - $\forall p, s, s'. \text{OnPath}(p, s) \land \text{OnPath}(p, s') \Rightarrow (s = s' \lor s < s' \land s' < s)$.
    - If $p \neq p'$ then there is situation that is on path $p$ but not on path $p'$:
      - $\forall p, p'. \forall s. (\text{OnPath}(p, s) \land \text{OnPath}(p', s') \Rightarrow p = p')$. 

- $\Sigma \cup D_{\text{path}}$:
  - $\forall s, a, s_2. (s = s_2 \land \text{Starts}(p_2, s_2)) \Rightarrow (\text{Starts}(p_1, s_1) \land \text{Suffix}(p_2, p_1, s_2))$.

7. Properties of Infinite Paths in the Sit.Calc. (III)
- Th-1. (Induction on paths):
  - $\Sigma \cup D_{\text{path}}$ entails that if some property $Q$ holds for all paths that start with an initial situation, and whenever $Q$ holds for all paths that start with situation $s$, then it holds for all paths that start with any successor situation to $s$, then the property $Q$ holds for all paths:
    - $\forall s, a. \text{Poss}(a, s) \land (\forall s'. \text{OnPath}(p, s) \land \text{OnPath}(p, s') \Rightarrow \exists p. \text{Suffix}(p, p, s))$.

8. Correctness of Proposed Axiomatization
- Idea: prove correctness by showing that paths defined by $\Sigma \cup D_{\text{path}}$, are isomorphic to path sequences
  - $\forall a$: mapping from natural numbers $N$ to situations; $\forall s$: an axiomatization of $N$
  - $\forall s, a. \text{Poss}(a, s)$ is an executable situation, and $\forall s$ there is an action $a$ such that:
    - $\forall a, s. \text{Poss}(a, s) \land (s + 1) = \text{do}(a, s)$
    - $\text{Matches}(p, a)$: there is a one-to-one mapping from the set of situations on path $p$ to the set of situations on path sequence $\sigma$
- Th-3. (Soundness): $\Sigma \cup D_{\text{path}} \Rightarrow (\exists s. \text{PathSeq}(\sigma) \land \text{Matches}(\sigma))$
- Th-4. (completeness): $\Sigma \cup D_{\text{path}} \Rightarrow (\forall s. \text{PathSeq}(\sigma) \Rightarrow \text{Matches}(\sigma))$.

9. Applications
- (Khan&Lespérance-10) discusses application to model temporally-extended prioritized goals and intentions
- (Khan&Lespérance-15) shows how CTL* formulae can be interpreted in the situation calculus and discusses other applications
  - e.g. can express that in situation $s$ there is a path over which $\Phi$ always holds, i.e. $E(\Phi)[s]$, using $\exists p. \text{Starts}(s, p) \land \forall s'. \text{OnPath}(p, s') \equiv \Phi(s')$

10. Conclusion and Future Work
- Introduced infinite paths in the sit. calc. along with axiomatization
  - all features of the sit. calc. are inherited and retained
  - allows first-order quantification over paths (and thus formulæ are much more readable)
  - Proved interesting properties
  - showed paths are well behaved and indeed correspond to an intuitive notion of paths
  - identified useful general properties for reasoning within the sit. calc. (e.g. induction on paths)
- Future Work:
  - utilize paths in the sit. calc. to develop/further refine applications involving infinite histories (e.g. execution semantics of non-terminating programs)