Finite-lattice approach to the O(2) and O(3) models in 1+1 dimensions and the (2+1)-dimensional Ising model

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The Lanczos scheme for finding low-lying eigenvalues of a sparse matrix of large dimension is applied to solving the Hamiltonian formulation for the O(2) and O(3) spin systems in $1+1$ dimensions and the Ising model in $2+1$ dimensions. We confirm results obtained for these models in other ways. The new method is shown to be competitive with the other methods available for solving these problems.

I. INTRODUCTION

In a previous paper,[1] to be called I, we used a new method[1,2] for solving lattice systems in the Hamiltonian formalism, looking toward future application to the gauge field theory, quantum chromodynamics (QCD), of interest in particle physics. Basically, the method consists of first solving the finite system exactly by the Lanczos technique for finding the low-lying eigenvalues of large sparse matrices and then invoking scaling arguments to extrapolate the results to the infinite-lattice limit. In I, we tested the method by applying it to the $Z(2)$ and $Z(3)$ models in $1+1$ dimensions and found that it works remarkably well. These two test cases, however, are rather special in three ways: They both have a finite number of states per site, they are both in $1+1$ dimensions, and they both have a conventional phase transition (a high-temperature disordered phase, a low-temperature ordered phase, and an algebraic vanishing of the mass gap at the transition point.) The first two points make the computation of the finite-system properties far less complicated, while the third point allows for a straightforward extrapolation to the infinite system.

In this paper, we test the method further by applying it to systems that do not have these features: the $(1+1)$-dimensional O(2) and O(3) models and the $(2+1)$-dimensional Ising model. The O(2) model has an infinite number of states per site and is expected to have a line of fixed points in its low-temperature phase. Furthermore, its mass gap is expected to have an essential zero at the transition point rather than an algebraic one. The O(3) model also has an infinite number of states per site but is expected to have a high-temperature phase only, with no phase transition at any finite temperature. The $(2+1)$-dimensional Ising model, on the other hand, is our first test case in dimensions higher than $1+1$.

We found that the finite-scaling arguments used in I do not generalize, in a simple way, to models having unconventional phase transitions. This, the *ad hoc* assumptions made in these arguments, and the ambiguity in choosing the scaling variables, as pointed out in I, all have led us to reformulate the finite-scaling theory in the renormalization-group language. In that language, one can understand the scaling assumptions at a deeper level and can arrive at the same results that finite scaling gives, in a natural and simple fashion. Furthermore, this new formulation can handle unconventional phase transitions such as that of O(2), by computing the β function directly, without resorting to extrapolation. This new formulation is presented in Sec. II.

The added complications due to having an infinite number of states per site require special considerations. These considerations are presented in Sec. III along with the results obtained. Aside from this, the method of calculation, i.e., the Lanczos scheme and the computer programming, was essentially identical to that used in I, to which we refer for details.

II. FINITE-SCALING THEORY: A RENORMALIZATION-GROUP APPROACH

We consider a finite-lattice system of N particles whose Hamiltonian is given by

$$H = (g/2a)W ,$$

where a is the lattice spacing. We will be using the following notation: The mass gap of H will be denoted by $G_H(g,N)$, that of W will be denoted by $G(g,N)$, and finally, we will denote the quantity $gG(g,N)$ by $\Lambda(g,N)$. Thus we have

$$G_H(g,N) = (g/2a)G(g,N) = (1/2a)\Lambda(g,N) . \quad (1)$$

Suppose that one diagonalizes the reduced Hamiltonian W exactly and obtains the functions $G(g,N)$ for several (but small) values of N. How can the critical properties of the $N=\infty$ system be inferred from these functions? There are two

ways to answer this question depending on how one interprets the functions $G(g,N)$.

A. The finite-system interpretation

Here one thinks of the functions $G(g,N)$ as the mass gaps of independent systems [independent in the sense that given $G(g,N)$, one cannot obtain $G(g,N')$ if $N\neq N'$], the only connection among them being that as $N\to\infty$, they approach the mass gap of the infinite system of interest. But since these functions were computed for fixed and small values of N, extrapolation arguments have to be used. These arguments (finite-scaling theory), originally due to Barber and Fisher,[2,3] were reviewed in I. If the $N=\infty$ system has a conventional phase transition, viz.,

$$G_H(g,\infty)\sim(g-g^*)^{\nu},\quad g\sim g^* \tag{2}$$

then these arguments give

$$G(g^*,N)=\text{const}\times N^{-1}, \tag{3}$$

$$\partial\Lambda(g,N)/\partial g|_{g=g^*}=\text{const}\times N^{-1+1/\nu}. \tag{4}$$

The first of these equations determines g^*, while the second determines ν, given g^*. This is how, in I, g^* and ν of the $Z(2)$ and $Z(3)$ models were determined.

B. The renormalization-group interpretation

Here we introduce a new interpretation of the renormalization group, slightly different from the usual block-spin approach to an infinite system. Our approach can be regarded as a further development of the phenomenological renormalization group of Sneddon and Stinchcombe,[4] but we feel we have made some important additions. We think of the functions $G(g,N)$ as describing the physics of different length scales of the same closed continuous system. One imagines starting with a field theory in one dimension on a circle of circumference L and then making it discrete to form a circular lattice. We imagine a circle, rather than a straight line, corresponding to the periodic boundary conditions used in computing $G(g,N)$. Such a closed (circular) system, described by periodic boundary conditions, shares some important properties with an infinite system, most importantly that there are no edges and hence no edge effects in the calculation. The process of converting to a lattice theory involves replacing the spatial derivatives in the theory by finite differences. These finite differences can be constructed with different values of the lattice spacing a or equivalently, the number of sites $N=L/a$. Note that L is fixed. By varying a, one obtains several pictures of the same physics taken with different resolutions. This, in effect, defines a renormalization-group (RG) transformation because for two systems with N and N' sites to have the same physics, we must have

$$G_H(g,N)=G_H(g',N'), \tag{5}$$

or, using Eq. (1) with $a=L/N$, $a'=L/N'$,

$$gNG(g,N)=g'N'G(g',N'),$$

which defines an effective RG transformation $g'=R(g)$. To clarify this point, consider the situation in Fig. 1(a) where $gNG(g,N)=2LG_H$, for the O(2) model, is plotted versus g for $N=2$–7. We see in the figure that the physics which corresponds to $2LG_H=2$ (say) can be described by a system of seven particles with coupling constant $g\approx 1.35$, or by a system of five particles with $g\approx 1.37$. Put differently, if one starts with $N=7$ and $g\approx 1.35$ and effects the RG transformation R defined by

$$R:\ a\to a'=ba,$$
$$N\to N'=N/b,$$
$$g\to g'=R(g),$$

where $b=\frac{7}{5}$, one will arrive at a system of five particles with $g\approx 1.37$ (see Fig. 2). In fact, with the aid of this simple RG interpretation, the entire phase structure of the model can be inferred from Fig. 1: If we start with $g>1$ [see Fig. 1(a)] and take $b>1$, we always end up with $g'>g$, which

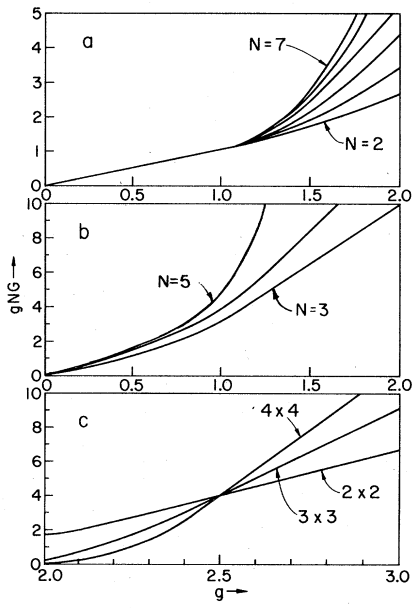

FIG. 1. The quantity $2LG(g,N)=gNG(g,N)$ is plotted versus g for (a) the O(2) model with $N=2,3,4,5,6,7$, (b) the O(3) model with $N=3,4,5$, and (c) for the Ising model with $N=2,3,4$, corresponding to lattice sizes 2×2, 3×3, and 4×4 [see Eq. (18)].

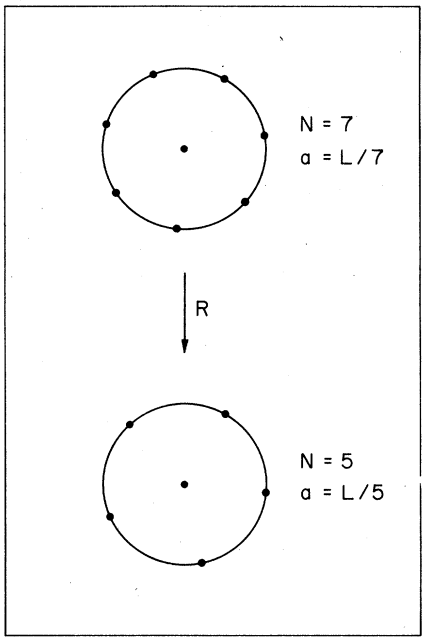

FIG. 2. This figure depicts a renormalization-group transformation R that maps a closed circular system of seven particles with coupling g onto a system of five particles with coupling g'.

implies that $g > 1$ is a high-temperature phase. On the other hand, if we start with $g \lesssim 1$, we end up with $g' = g$, indicating a line of fixed points extending from $g = 0$ to $g \approx 1$. Similarly, we see from Fig. 1(b) [which is the same as (a) but for the O(3) model with $N = 3-5$] that O(3) resides in the high-temperature phase for all g. Figure 1(c) is the same as (a) but for the $(2+1)$-dimensional Ising model with lattice sizes 2×2, 3×3, and 4×4. The figure indicates a high-temperature phase ($g' > g$) and a low-temperature phase ($g' < g$) separated by a fixed point ($g' = g$) at $g \approx 2.5$.

To make these arguments quantitative, assume for now that the model of interest has a conventional phase structure so that Eq. (2) holds. In that case, we can expand both sides of Eq. (5) near g^*:

$$G_H(g,N) = G_H(g',N'), \quad g' = R(g)$$
$$G_H(g^*,N) = G_H(R(g^*),N'), \tag{6}$$

$$(\partial G_H/\partial g)|_{g=g^*} = (\partial G_H/\partial g')|_{g'=g^*}(dR/dg)|_{g=g^*}. \tag{7}$$

These two equations are the same as Eqs. (3) and (4). Starting from Eq. (6) and using Eq. (1) (with $a = L/N$, $a' = L/N'$) and the fact that $R(g^*) = g^*$, by definition of g^*, we easily derive Eq. (3). Furthermore, if we recall from renormalization-group theory that the thermal index y is defined by

$$(dR/dg)|_{g=g^*} = b^y,$$

Eq. (7) becomes (with $b = N/N'$)

$$N^{-y}(\partial G_H/\partial g)|_{g=g^*} = N'^{-y}(\partial G_H/\partial g')|_{g'=g^*},$$

which reduces to Eq. (4) if we use Eq. (1) and the fact that $\nu = 1/y$. We see then that the finite-scaling predictions [Eqs. (3) and (4)] can be easily derived from simple renormalization ideas applied to a closed continuous system, when the phase transition is conventional.

We are now going to present a new technique for extracting the critical properties of the infinite system of interest. This technique is sufficiently general that it can handle situations in which the phase transition is not characterized by an algebraic zero of the mass gap. The idea is to compute the β function of the closed (circular) system directly, without resorting to extrapolation in N. The Callan-Symanzik β function is defined by

$$\beta(g) = a(\partial g/\partial a)|_{G_H = \text{const}}. \tag{8}$$

In the present interpretation $a = L/N$ arises from making a closed continuous system of fixed length L discrete, so changes of a are related to changes in N, $da/a = -dN/N$. But there is also N dependence in $G(g,N)$ in addition to a trivial overall factor of N, namely $G_H = NgG(g,N)/2L$. Setting $G_H = \text{constant}$ and using these relations we find

$$(\partial G_H/\partial g)dg + (\partial G_H/\partial N)dN = 0,$$

which leads to

$$\beta(g)/g = (N\partial G_H/\partial N)/(g\partial G_H/\partial g) \tag{9a}$$

$$= (\partial \ln G_H/\partial \ln N)/(\partial \ln G_H/\partial \ln g) \tag{9b}$$

$$= \frac{\partial \ln[NgG(g,N)]/\partial \ln N}{\partial \ln[NgG(g,N)]/\partial \ln g} \tag{9c}$$

$$= \frac{\ln[N'G(g,N')/NG(g,N)]}{\ln(N'/N)\{1+\frac{1}{2}g(\partial/\partial g)\ln[NG(g,N)N'G(g,N')]\}}. \tag{9d}$$

Here the form (9d) is obtained by approximating derivatives in N by finite differences. This is the form actually used later in the numerical calculations.

It is important to note that the formulas (9) are different from those often used because of our interpretation's dependence on the discrete version of a continuous closed system of fixed length L. The usual formula for an infinite open system, used for example in Ref. 5, is obtained by replacing $G(g,N)$ by its $N\to\infty$ limit $G(g)$, which is independent of N. From Eq. (9c) one obtains in this way

$$\beta(g)/g = 1/(\partial \ln\Lambda/\partial \ln g), \quad \Lambda = gG(g)$$
$$= G(g)/[G(g) + g\partial G(g)/\partial g]. \tag{10}$$

Another possibility would be to use Eq. (10) with the $N\to\infty$ limit $G(g)$ replaced by $G(g,N)$ for finite N. A β function calculated this way displays properties of a finite system, for example it never vanishes, corresponding to the absence of phase transitions for a finite system. On the other hand, our β function (9), which includes proper account of the derivatives $\partial G(g,N)/\partial N$, has the properties of the β function of an infinite system, and thus does vanish at a transition point. We find that numerical values obtained from Eq. (9d) are nearly independent of N and N', even for very small values of these numbers. Naturally, the closest approximation to an infinite open system will be obtained for the largest values of N, N' that one can force the computer to calculate.

Using Eq. (9d) one can compute $\beta(g)/g$ as a function of g and then fit it to the expected behavior of the $N=\infty$ β function. In the case of a conventional phase transition, one expects for the $N=\infty$ β function the behavior

$$\beta(g) = (g - g^*)/\nu, \quad g \sim g^*$$

[obtained by substituting $\Lambda = \text{const}\times(g-g^*)^\nu$ in Eq. (10)] and by fitting the β of Eq. (9d) to this, one can infer g^* and ν. In the case of O(2), however, where the phase transition is not conventional, one would fit β of Eq. (9d) to[5,6]

$$\beta(g) \sim (g - g^*)^{1+\sigma}, \quad g \sim g^*$$

(obtained by substituting $\Lambda = \text{const} \times \exp[-c(g-g^*)^{-\sigma}]$ in Eq. (10)) and thus determine g^* and σ.

III. APPLICATIONS

A. The (1 + 1)-dimensional O(2) model

1. Theory

The Hamiltonian of the model is given by[5]

$$H = (g/2a)\sum_{i=1}^{N}\{J_z^2(i) - \tfrac{1}{2}x[J_+(i)J_-(i+1) + \text{H.c.}]\},$$

where $x = 2/g^2$ and $J_\pm(N+1) = J_\pm(1)$. The operators J_z, J_+, and J_- are the usual angular momentum operators on two-dimensional angular momentum states $|M\rangle$:

$$|M\rangle \to \frac{1}{\sqrt{2\pi}} e^{iM\varphi}, \quad 0 < \varphi < 2\pi$$

$$J_z|M\rangle = M|M\rangle,$$

$$J_\pm|M\rangle = |M \pm 1\rangle.$$

The model is expected[5,6] to have a line of fixed points extending from $x=\infty$ to $x=x^*$, with $x^* = 1.85 \pm 0.15$.[5] Furthermore, using the notation of Sec. II, the gap is supposed to have the form

$$G_H(g) \sim \exp[-c(g-g^*)^{-\sigma}].$$

The β function of the system is thus thought to have the behavior

$$\beta(g) \equiv 0 \quad \text{for } g < g^*, \tag{11}$$

$$\beta(g) \sim (g-g^*)^{1+\sigma} \quad \text{for } g \to g^{*+}, \tag{12}$$

$$\beta(g) = g(1 - 2x + 2.5x^2 - 3.0625x^3 + \cdots) \quad \text{for } g \gg g^*, \tag{13}$$

with $\sigma = 0.7 \pm 0.1$ (Ref. 5) (while Ref. 6 predicts $\sigma = 0.5$).

2. Computation

We see that the single-site operator $J_z^2(i)$ has an infinite number of states per site with eigenvalues M^2 ($M = 0, \pm1, \pm2, \ldots$). The Lanczos technique of I, however, requires this number of states to be finite. To overcome this problem we had to put a cutoff ($M_{\max}$) on M such that for all sites $|M| \le M_{\max}$. And in order to ensure that the final answer is cutoff independent, we kept increasing the value of $M_{\max}$ until the answer became stable (to within 0.1%). The calculation was done for $N = 2, 3, 4, 5, 6, 7$. A typical situation is shown in Fig. 3, where $G(g,N)$ is plotted versus x for $N=5$. We see in that figure that for $M_{\max} \ge 4$, $G(g,N)$ becomes independent of $M_{\max}$. It is important to notice, however, that one has to increase $M_{\max}$ even further if $G(g,N)$ is to be computed for $x > 10$. This is due to the fact that in the Lanczos method, one

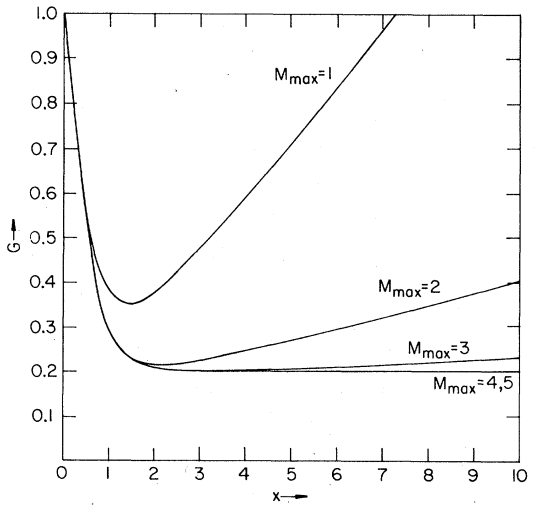

FIG. 3. The O(2) mass gap $G(g,N)$ is plotted against $x = 2/g^2$ for $N=5$ and for increasing values of the cutoff $M_{\max}$.

has to iterate more times, thus introducing larger M values for x large, before the eigenvalues converge to some given accuracy. $M_{\max}$ determines the number of bits necessary to represent a state. If we reserve l bits per site, in each memory word, then we must pick l such that $2^l > 2M_{\max} + 1$. The final values of $G(g,N)$ are plotted in Fig. 4 versus x for $N=2$–7. We notice in Fig. 4 that for large x, G approaches $1/N$. This observation is discussed in the Appendix. We found that 15 Lanczos iterations were sufficient to obtain the eigenvalues to six significant figures and the gap to three significant figures. The maximum number of linear combinations in the Lanczos states that we had to deal with was ~9000, which occurs for $N=7$ and $M_{\max}=3$.

3. *Results*

The location of the transition point x^* can be determined from Eq. (3):

$$G(g^*,N) = \text{const} \times N^{-1}$$

or (14)

$$\ln G(g^*,N) = \text{const} - \ln N .$$

Figure 5 shows a log-log plot of G versus N for several values of $x = 2/g^2$. We see from Fig. 5 that for $x \geq 2$, we obtain a straight line of slope -1. This means Eq. (14) is satisfied for all these values of x, which indicates that we have a line of fixed points extending from $x=\infty$ to $x \approx 2$. This situation is to be contrasted with that of Fig. 3 and Fig. 4 of I in which Eq. (14) is satisfied for one value of x only. We estimate from this analysis $x^* = 1.9 \pm 0.2$. To obtain σ, we use Eq. (9d) to compute the β function. The results are shown in Fig. 6 for $N=2$, $N'=3$ and $N=6$, $N'=7$. (The β functions computed from other possible pairs of N values lie between the two curves shown in Fig. 6). We see from Fig. 6 that our β function vanishes for all $g \lesssim 1$, in agreement with Eq. (11); furthermore, it approaches the high-temperature behavior [Eq. (13)] for $g > 2.5$. Finally, if we fit the β function to the behavior given in Eq. (12), we obtain a good fit with $g^* = 0.998 \pm 0.001$, or $x^* = 2.006$ [in agreement with the analysis based on Eq. (14)], and $\sigma = 0.51 \pm 0.01$.

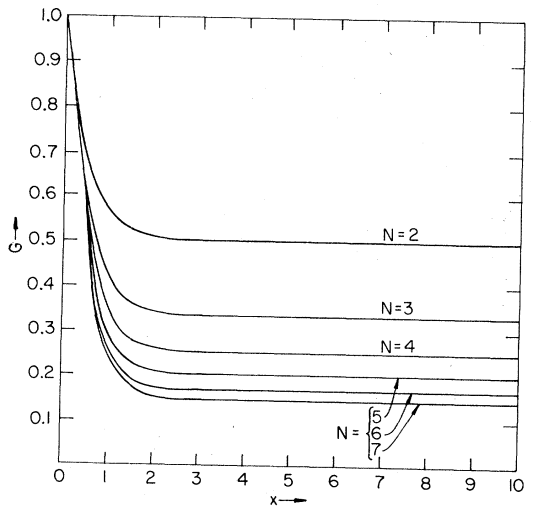

FIG. 4. The mass gaps $G(g,n)$ of the O(2) model are plotted versus $x = 2/g$ for $N = 2, 3, 4, 5, 6, 7$.

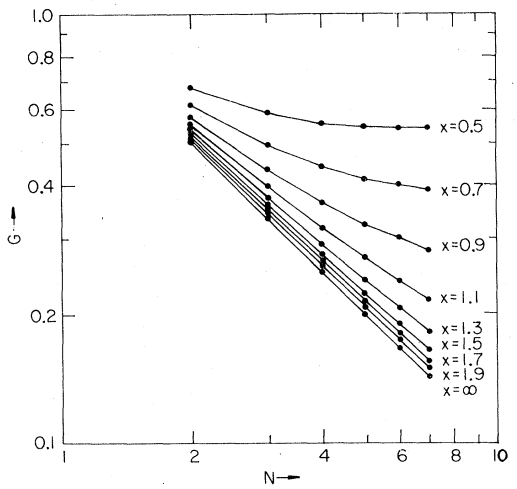

FIG. 5. A log-log plot of $G(g,N)$ of the O(2) model against N for several $x = 2/g^2$ values. A straight line of slope -1 obtains when $x > 2$.

B. The (1 + 1)-dimensional O(3) model

1. *Theory*

The Hamiltonian of the model is given by[5]

$$H = (g/2a) \sum_{i=1}^{N} [J^2(i) - x \cos\Theta_{i,i+1}] ,$$

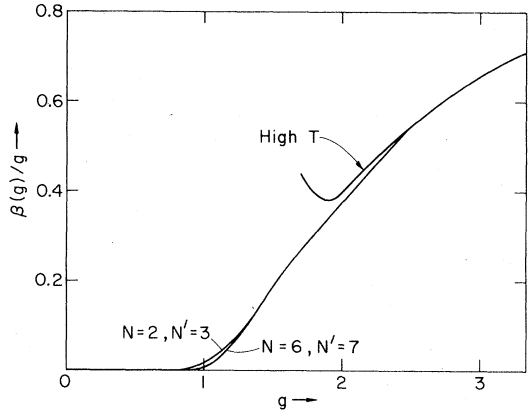

FIG. 6. The O(2) β functions corresponding to $N=5$, $N'=6$ and $N=6$, $N'=7$ [see Eq. (9d)] are plotted versus g along with the high-temperature behavior of Eq. (13).

where $x = 2/g^2$. The operator $\vec{J}$ is the usual angular momentum operator on three-dimensional orbital angular momentum states $|L,M\rangle$:

$$|L,M\rangle \rightarrow Y_L^M(\theta,\varphi)\,,$$
$$J^2|L,M\rangle = L(L+1)|L,M\rangle\,,$$
$$J_z|L,M\rangle = M|L,M\rangle\,.$$

The quantity $\cos\Theta_{i,i+1}$ is the cosine of the angle between unit vectors in directions $\theta_i\varphi_i$ and $\theta_{i+1}\varphi_{i+1}$ at adjacent sites. The effect of this operator on the basis states is obtained using the spherical harmonics addition theorem and the Clebsch-Gordan series

$$\cos\Theta_{i,i+1} = \frac{4\pi}{3}\sum_M (-1)^M Y_1^M(\theta_i,\varphi_i) Y_1^{-M}(\theta_{i+1},\varphi_{i+1})\,,$$

$$(4\pi/3)^{1/2} Y_{L_i}^{M_i}(\theta_i,\varphi_i) Y_1^M(\theta_i,\varphi_i) = [(L_i+1)/(2L_i+3)]^{1/2} C(L_i,1,L_i+1;M_i,M) Y_{L_i+1}^{M_i+M}(\theta_i,\varphi_i)$$
$$-[L_i/(2L_i-1)]^{1/2} C(L_i,1,L_i-1;M_i,M) Y_{L_i-1}^{M_i+M}(\theta_i,\varphi_i)\,,$$

where the $C(L_i,1,L_i\pm1;M_i,M)$ are Clebsch-Gordan coefficients for which simple formulas are available.[7]

The model is expected[5] to reside in a disordered phase for all g. The β function is expected to have the high-temperature behavior[5]

$$\beta(g)/g = 1 - 0.66x + 0.2963x^2 - 0.122\,84x^3 + \cdots\,, \tag{15}$$

and the low-temperature behavior[8]

$$\beta(g)/g = g/2\pi + (g/2\pi)^2 + \cdots\,. \tag{16}$$

2. Computation

Here, again, the single-site operator $J^2(i)$ has an infinite number of states with eigenvalues $L(L+1), L = 0, 1, 2, \ldots$. As in Sec. III A 2, we introduce a cutoff ($L_{\max}$) and keep increasing it until the answer stabilizes. We found that for $x \leq 10$, $L_{\max} = 4$ was sufficient to make $G(g,N)$ independent of $L_{\max}$. The calculation was done for $N = 3, 4, 5$. The results are shown in Fig. 7, where $G(g,N)$ is plotted against x for $N = 3$–5. We found that 10 Lanczos iterations were sufficient to obtain the eigenvalues to six significant figures and the gap to three significant figures. The maximum number of linear combinations in the Lanczos states needed was ~15 000 for $N = 5$ and $L_{\max} = 3$.

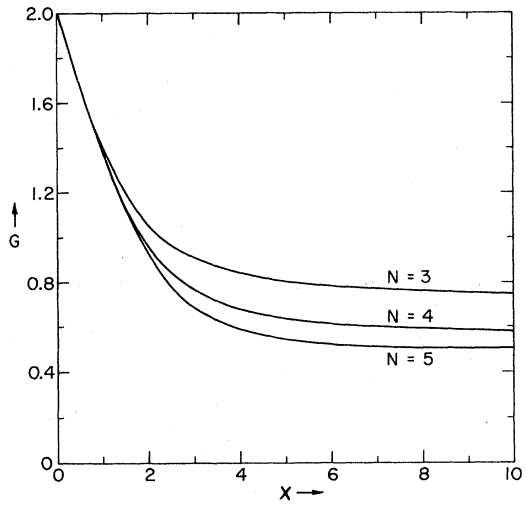

FIG. 7. The mass gaps $G(g,N)$ of the O(3) model are plotted versus $x = 2/g^2$ for $N = 3, 4, 5$.

3. Results

The β function of the model, computed from Eq. (9d), is shown in Fig. 8 for $N = 3, N' = 4$ and $N = 4, N' = 5$, along with the high-temperature [Eq. (15)] and the low-temperature [Eq. (16)] series.

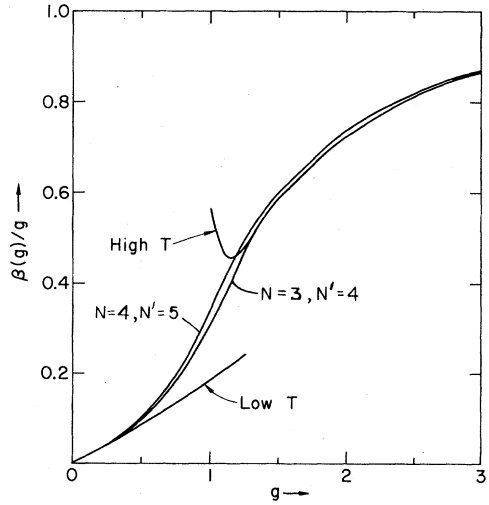

FIG. 8. The O(3) β functions corresponding to $N = 3$, $N' = 4$ and $N = 4$, $N' = 5$ [see Eq. (9d)] are plotted versus g along with the high- and low-temperature behaviors of Eqs. (15) and (16).

Our β function is seen to interpolate the high- and low-temperature series remarkably well. The nonvanishing of β indicates an absence of a phase transition for all values of g.

C. The (2+1)-dimensional Ising model

1. Theory

The Hamiltonian of the model is given by

$$H=(g/2a)\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\{\sigma_z(i,j)-x\sigma_x(i,j)\times[\sigma_x(i+1,j)+\sigma_x(i,j+1)]\}.$$

We use periodic boundary conditions. The point $i,N+1$ is identified with $i,1$ for all i and the point $N+1,j$ is identified with $1,j$ for all j. Thus in the Hamiltonian

$$\sigma_x(i,N)\sigma_x(i,N+1)=\sigma_x(i,N)\sigma_x(i,1)$$

and

$$\sigma_x(N,j)\sigma_x(N+1,j)=\sigma_x(N,j)\sigma_x(1,j).$$

This model is expected to have a conventional phase transition

$$G_H\sim(g-g^*)^{\nu},\quad g\sim g^*$$
$$\beta(g)=(g-g^*)/\nu, \tag{17}$$

with $\nu=0.638\pm0.002$.[9] (As far as we know, there is no treatment of this model in the Hamiltonian formulation, and thus no value of g^* is available.)

2. Computation

The calculation was done for lattice sizes $M_1\times M_2$ with $M_1,M_2=2$–4. The variable N that was used in Sec. II is the linear dimension of the system and for this model, one could choose $N=(M_1M_2)^{1/2}$, for example. We found that the numbers scale nicely if one used

$$N=\sqrt{2}M_1M_2/(M_1^2+M_2^2)^{1/2}, \tag{18}$$

as advocated in Ref. 10. The storage of the states is much easier than for the previous two models since each site can be in one of two states only (up or down). Thus one reserves one bit for each site. In fact, since on CDC computers one has sixty bits per word, one can represent the state of up to 7×7 lattice in one word. The maximum number of linear combinations in the Lanczos states needed reached ~1100 for $M_1=4,M_2=4$. The functions $G(g,N)$ are plotted in Fig. 9 against x for several values of M_1 and M_2.

3. Results

The location of the fixed point x^* can be determined, as in Sec. III A 3, from Eq. (3). Figure 10 is a log-log plot of $G(g,N)$ versus N for several x values. We see that a straight line of slope -1 obtains when $x\approx0.32$. We estimate from this analysis $x^*=0.320\pm0.001$ or in terms of g, $g^*=2.500\pm0.004$. The index ν can be computed with the aid of Eq. (4). We obtain $\nu=0.65$. g^* and ν can also be obtained, independently, from the β function. Figure 11 shows the β function of this model, computed from Eq. (9d), as a function of g, for $N=2,N'=3$ and $N=3,N'=4$. (All the other cases give β functions lying in between.) We fitted these β functions to the behavior given in Eq. (17). A good fit obtains when $g^*=2.49\pm0.02$ ($x^*=0.322\pm0.003$) and $\nu=0.640\pm0.004$, in agreement with the analysis based on Eqs. (3) and (4).

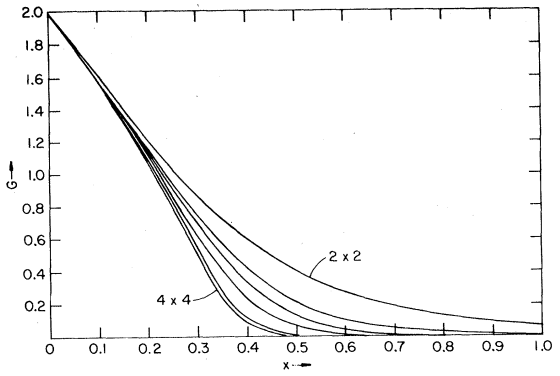

FIG. 9. The mass gaps $G(g,N)$ of the Ising model are plotted against $x=2/g^2$ for lattice sizes 2×2, 2×3, 2×4, 3×3, 3×5, and 4×4.

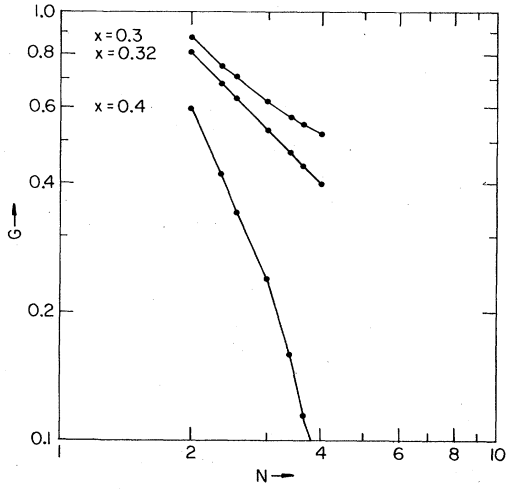

FIG. 10. A log-log of $G(g,N)$ versus N for the Ising model with lattice sizes 2×2, 2×3, 2×4, 3×3, 3×4, 3×5, and 4×4. The corresponding N values are computed from Eq. (18).

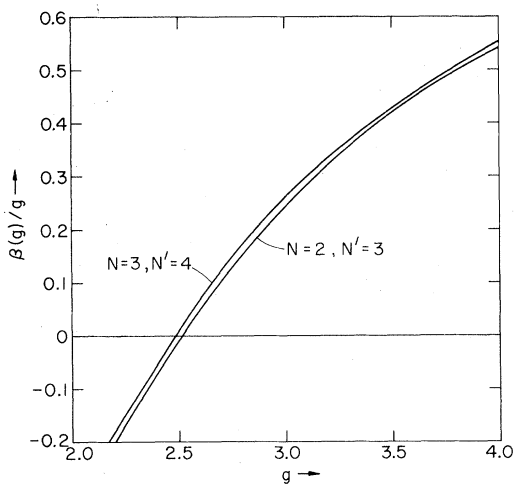

FIG. 11. The Ising β functions corresponding to $N=2$, $N'=3$ and $N=3$, $N'=4$ [see Eq. (9d)], versus g.

IV. CONCLUSION

The Lanczos technique introduced in I and the RG interpretation presented in this paper were applied to a variety of spin systems, and have led to the following results: The $(2+1)$-dimensional Ising model is shown to have an ordinary phase transition at $x^*=0.32$, with a correlation length index of 0.640, slightly above the high-temperature series value 0.638. For the $(1+1)$-dimensional O(2) model we confirm the results of the Kosterlitz renormalization group, namely a line of fixed points extending from $x=\infty$ to $x=x^*$, an essential zero for the mass gap at x^*, and $\sigma=0.5$. Furthermore, our value for $x^*=2.0$ is consistent with that of Ref. 5. Our results for O(3) in $1+1$ dimensions rule out the possibility of a phase transition at any finite temperature. Furthermore, our β function interpolates very nicely between the high- and low-temperature series results.

These results, along with those obtained in I for the $(1+1)$-dimensional $Z(2)$ and $Z(3)$ models, lead us to believe that the method presented here is quite competitive both qualitatively and quantitatively with other methods used to study these problems, most notably high- and low-temperature expansions and Monte Carlo simulations. In fact, our method has certain important advantages. High- and low-temperature expansions give very good results in one limit or the other, which must then be extrapolated (by Padé techniques) to the region of interest. The method used here is equally valid for all values of the coupling constant. On the other hand, we do not have to cope with the sort of statistical errors inherent in Monte Carlo methods. Further research is necessary to determine if the method can be made to work for the gauge field theory QCD of current interest in particle physics. Such an investigation is underway.

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APPENDIX

In Sec. III we found numerically that the O(2) model reduced Hamiltonian for N sites

$$W=\sum_{i=1}^{N}\left[-\frac{\partial^2}{\partial\varphi_i^{\,2}}-x\cos(\varphi_i-\varphi_{i+1})\right]$$

leads to a simple result for the gap $G(x,N)$ in the large-x limit:

$$G(x,N)=1/N,\quad x\to\infty. \tag{A1}$$

The origin of this result appears to be a "center-of-mass" effect.

Tranforming to center-of-mass and relative coordinates

$$\psi=(\varphi_1+\varphi_2+\cdots+\varphi_N)/N,$$
$$\theta_1=\varphi_2-\varphi_1,$$
$$\theta_2=\varphi_3-\varphi_2,$$
$$\cdots$$
$$\theta_{N-1}=\varphi_N-\varphi_{N-1},$$

the reduced Hamiltonian becomes

$$W=W'-(1/N)\partial^2/\partial\psi^2,$$
$$W'=\sum_{i=1}^{N-1}\left(-2\frac{\partial^2}{\partial\theta_i^{\,2}}-x\cos\theta_i\right)-2\sum_{i=1}^{N-2}\frac{\partial^2}{\partial\theta_i\partial\theta_{i+1}}-x\cos\left(\sum_{i=1}^{N-1}\theta_i\right).$$

The operator W has eigenfunctions of the form

$$\Psi=e^{iM\psi}F(\theta_1,\theta_2,\ldots,\theta_{N-1}). \tag{A2}$$

Since Ψ is periodic with period 2π in each of the original variables φ_i, its value is unchanged if we increase each and every φ_i by 2π. Performing this manipulation on Eq. (A2) shows that M must be an integer. The eigenvalues of W are thus of the form

$$E=E'+M^2/N, \tag{A3}$$

where E' is an eigenvalue of the complicated op-

erator W', the corresponding eigenfunction being $F(\theta_1, \theta_2, \ldots, \theta_{N-1})$.

For x large, the operator W can be simplified by expanding all cosines, $\cos\theta \to 1 - \theta^2/2$. This leads to a problem of coupled harmonic oscillators. For large x the separations between energy levels of these oscillators will be proportional to $x^{1/2}$. Thus, the two lowest levels of the system will occur when all the oscillators are in their ground states and $M = 0$ or $M = 1$. We thus deduce from Eq. (A3) the behavior given by Eq. (A1) for the gap in the large-x limit.

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