Computing Probabilistic Bisimilarity Distances via Policy Iteration

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(joint work with Franck van Breugel)

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1 Probabilistic Bisimilarity Distances
   - Labelled Markov Chains (LMCs)
   - Behavioural Pseudometrics
   - Couplings

2 Computation of Bisimilarity Distances
   - Algorithm: BBLM
   - Simple Stochastic Games (SSGs)
   - From LMCs to SSGs
   - BBLM Algorithm is Exponential
Probabilistic Bisimilarity Distances

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A labelled Markov chain is a tuple $\langle S, L, \tau, \ell \rangle$ consisting of
- a finite set $S$ of states,
- a finite set $L$ of labels,
- a rational transition function $\tau : S \rightarrow \text{Dist}(S)$, and
- a labelling function $\ell : S \rightarrow L$. 

![Diagram of a labelled Markov chain](image-url)
Fundamental problem

Behavioural equivalences are not robust for systems with real-valued data.
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Robust alternative

Instead of an equivalence relation

\[ \sim : S \times S \rightarrow \{\text{True, False}\} \]

use a pseudometric

\[ d : S \times S \rightarrow [0, 1]. \]
Probabilistic bisimilarity distance

The smallest $d : S \times S \to [0, 1]$ satisfying

$$d(s, t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \min_{\omega \in \Omega(\tau(s), \tau(t))} \sum_{u, v \in S} \omega(u, v) d(u, v) & \text{otherwise} \end{cases}$$

A coupling of probability distributions $\mu$ and $\nu$ on $S$ is a probability distribution $\omega$ on $S \times S$ with marginals $\mu$ and $\nu$, that is, for all $u, v \in S$,

$$\sum_{v \in S} \omega(u, v) = \mu(u)$$

$$\sum_{u \in S} \omega(u, v) = \nu(v)$$

The set of couplings of $\mu$ and $\nu$ is denoted by $\Omega(\mu, \nu)$. 
Coupling

\[ \begin{align*}
  s & \xrightarrow{0.3} s_2 \\
  s & \xrightarrow{0.2} s_3 \\
  s_2 & \xrightarrow{0.5} s_1 \\
  s_3 & \xrightarrow{0.5} s_2 \\
  t & \xrightarrow{0.5} t_1 \\
  t_1 & \xrightarrow{0.5} t_2 \\
  t_2 & \xrightarrow{0.5} t
\end{align*} \]
The cost transporting from $s$ to $t$ is larger than 0.
The cost transporting from $s$ to $t$ is 0.
Probabilistic Bisimilarity distance

The smallest $d : S \times S \rightarrow [0, 1]$ satisfying

$$d(s, t) = \begin{cases} 
1 & \text{if } \ell(s) \neq \ell(t) \\
\min_{\omega \in \Omega(\tau(s), \tau(t))} \sum_{u, v \in S} \omega(u, v) d(u, v) & \text{otherwise}
\end{cases}$$


Proposition (DGJP 1999)

$s \sim t$ if and only if $d(s, t) = 0$. 
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Algorithm to compute the bisimilarity distances

Algorithm to compute the bisimilarity distances


BBLM algorithm = basic algorithm + optimization

this talk + on-the-fly
Anne Condon was the first to study simple stochastic games from a computational point of view in 1992.
The value of a vertex is the probability that the max player wins the game (reaches 1) provided that both players use optimal strategies (the min player tries not to reach 1).
Definition

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Values of a SSG

Definition

The *value* of a vertex is the probability that the max player wins the game (reaches 1) provided that both players use optimal strategies (the min player tries not to reach 1).
For each LMC we construct a corresponding SSG such that

<table>
<thead>
<tr>
<th>LMC</th>
<th>SSG</th>
</tr>
</thead>
<tbody>
<tr>
<td>distance</td>
<td>value</td>
</tr>
<tr>
<td>BBLM algorithm</td>
<td>simple policy iteration</td>
</tr>
</tbody>
</table>

Ronald Howard introduced policy iteration in 1958.
With every pair of states \((s, t)\) of the LMC we associate a vertex of the SSG.

- If \(\ell(s) \neq \ell(t)\) then \(d(s, t) = 1\).
- If \(s \sim t\) then \(d(s, t) = 0\).
Otherwise,

$$d(s, t) = \min_{\omega \in V(\Omega(\tau(s), \tau(t)))} \sum_{u, v \in S} \omega(u, v) d(u, v)$$
Otherwise,

\[ d(s, t) = \min_{\omega \in V(\Omega(\tau(s), \tau(t)))} \sum_{u,v \in S} \omega(u, v) d(u, v) \]
BBLM algorithm is exponential

Simple policy iteration

choose a random initial policy
while exists some vertex not locally optimal
    adjust the policy at that vertex
Simple policy iteration

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Theorem

For each $n \in \mathbb{N}$, there exists an LMC of size $O(n)$ such that simple policy iteration takes $\Omega(2^n)$ iterations.
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Theorem

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Proof idea: Implement an “$n$-bit counter.”
BBLM algorithm is exponential

We start with the following LMC.
BBLM algorithm is exponential

The LMC corresponds to the following SSG.
BBLM algorithm is exponential
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Our results:

- The basic BBLM algorithm is simply policy iteration.
  - To define the SSG we need to decide probabilistic bisimilarity.

- In the worst case, the (basic) BBLM algorithm is exponential.

- In practice, the (basic) BBLM algorithm performs much better than all other algorithms.
Future work

- Is the worst-case running time of general policy iteration exponential?
- What is the expected running time of policy iteration if the selection is randomized?
- Can policy iteration be applied to compute the bisimilarity distances of probabilistic automata? If it can, what is the time complexity? If it cannot, why not?