

# Bisimulation for Markov Decision Processes Through Families of Functional Expressions

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**Abstract.** We transfer a notion of quantitative bisimilarity for labelled Markov processes [1] to Markov decision processes with continuous state spaces. This notion takes the form of a pseudometric on the system states, cast in terms of the equivalence of a family of functional expressions evaluated on those states and interpreted as a real-valued modal logic. Our proof amounts to a slight modification of previous techniques [2, 3] used to prove equivalence with a fixed-point pseudometric on the state-space of a labelled Markov process and making heavy use of the Kantorovich probability metric. Indeed, we again demonstrate equivalence with a fixed-point pseudometric defined on Markov decision processes [4]; what is novel is that we recast this proof in terms of integral probability metrics [5] defined through the family of functional expressions, shifting emphasis back to properties of such families. The hope is that a judicious choice of family might lead to something more computationally tractable than bisimilarity whilst maintaining its pleasing theoretical guarantees. Moreover, we use a trick from descriptive set theory to extend our results to MDPs with bounded measurable reward functions, dropping a previous continuity constraint on rewards and Markov kernels.

## 1 Introduction

Probabilistic bisimulation is a notion of state-equivalence for Markov transition systems, first introduced by Larsen and Skou [6] based upon bisimulation for nondeterministic systems by Park and Milner [7, 8]. Roughly, states are deemed equivalent if they transition with the same probability to classes of equivalent states.

In the context of labelled Markov processes (LMPs), a robust quantitative notion of probabilistic bisimilarity has been devised in the form of a class of behavioural pseudometrics, or *bisimilarity metrics*, defined on the state space of a given process [9, 1, 10, 2, 11]. The defining characteristic of these metrics is that the kernel of each is probabilistic bisimilarity, and otherwise each assigns a distance between 0 and 1 that measures the degree to which two states are bisimilar.

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Bisimilarity metrics were initially defined in terms of a family of functional expressions interpreted as a real-valued logic on the states of an LMP by Desharnais et al. [9], building on ideas of Kozen [12] that logic could be generalized to handle probabilistic phenomena. Subsequently, van Breugel and Worrell [2] used category theory to define another class of bisimilarity metrics, and showed that their definition was equivalent to a slightly modified version of the metrics of Desharnais et al. in terms of the family of functional expressions considered. Desharnais et al. [1] in turn reformulated this latter version solely in terms of order-theoretic fixed-point theory. A crucial component in these formulations was the recognition that the initial version, when lifted to a metric on distributions, could be expressed as a Kantorovich metric.

For finite systems, the various formulations readily admit a variety of algorithms to estimate the distances [9, 3, 13]. In particular, the initial formulation in terms of a family of functional expressions led to an exponential-time algorithm based on choosing a suitable finite sub-family of functionals. This was vastly improved in [3] wherein an iterative polynomial-time algorithm exploited the fact that the Kantorovich metric is a specialized linear program.

In [14, 15, 4], the fixed-point presentation of the LMP bisimilarity metric was adapted to finite Markov decision processes (MDPs) and MDPs with continuous state spaces, bounded continuous rewards, and (weakly) continuous Markov kernels. Insofar as finite systems are concerned, the addition of the reward parameter is minor; in fact, the iterative polynomial-time algorithm applies more or less directly [14]. Unfortunately, even for very simple toy systems the experimental space and time required is too great to be of practical use. This issue was explored in [16] where a Monte Carlo approach to estimating the Kantorovich metric, and the bisimilarity metric for MDPs in general, was devised and shown to outperform other approaches in an experimental setting. The Monte Carlo approach was even extended to MDPs in which the state space is a compact metric space [4]. However, this line of investigation is still very preliminary.

In this work, we seek to complete further the picture of bisimilarity metrics for MDPs by presenting a family of functional expressions that induce the fixed-point bisimilarity metric, in analogy with the results of [2] for LMPs. We aim to shift the study of equivalences on MDP states to the study of such families and their properties. The right choice of family might lead to a more easily computable equivalence whilst maintaining some important theoretical guarantees. Additionally, we hope further to investigate Monte Carlo approaches to estimating similarity, for example, by sampling from a given family of functions. More specifically, we carry out the following.

1. We adapt Proposition 2 of [2] to MDPs, showing that a class of functional expressions interpreted as a real-valued logic on the states of a given MDP is dense in the class of Lipschitz functions with respect to the pseudometric induced by the family. It is important to note that the proof here appears almost unchanged from [2]; what is important is that we recast the result in the terminology and conceptual framework of integral probability metrics and their generating classes of functions.
2. We remove the continuity constraints of Theorem 7, which establishes the fixed-point bisimilarity metric for continuous MDPs, using techniques from descriptive set theory. This is, to the best of our knowledge, an original result.
3. We propose a preliminary Monte Carlo technique for estimating the bisimilarity metric by sampling from the family of functional expressions that encodes bisimilarity for MDPs. This too, appears to be an original result, but is based on a heuristic method and experimental evidence presented in [17].

The paper is organized as follows. In Section 2, we provide a brief summary on the relevant development of bisimilarity metrics and related results for LMPs and MDPs. In Section 3, we establish

a family of functional expressions that induces a metric equal to a previously-defined bisimilarity metric for MDPs, and then generalize the applicability of this result by removing previous continuity constraints. Finally, in Section 4 and Section 5, we propose a Monte Carlo method for approximating the bisimilarity metric by sampling from a family of functional expressions, and conclude with suggestions for future work.

## 2 Background

The purpose of this section is to recall the development of pseudometrics capturing bisimilarity for labelled Markov processes, and set down what has already been carried over to Markov decision processes. In doing so, we fix some basic terminology and notation for probabilistic systems. Unless otherwise stated, all the results of this section can be found in Prakash’s book on labelled Markov processes [18].

### 2.1 Probability Measures on Polish Metric Spaces

Since we deal primarily with uncountably infinite state spaces, we must take into account the tension involved in imposing the right amount of structure on a space for general theoretical utility and imposing the right amount of structure for practical application. Much of the work on labelled Markov processes has been cast in the setting of Polish spaces and analytic spaces, which are general enough to include most practical systems of interest while structured enough to admit many useful theorems.

#### Definition 1.

1. A Polish metric space is a complete, separable metric space.
2. A Polish space is a topological space that is homeomorphic to a Polish metric space.
3. An analytic set is a subset of a Polish space that is itself the continuous image of a Polish space.
4. A standard Borel space is a measurable space that is Borel isomorphic to a Polish space.

If  $(X, \tau)$  is a topological space, then  $\mathbb{C}^b(X)$  is the set of continuous bounded real-valued functions on  $X$ . If  $(X, \mathcal{B}_X)$  is a standard Borel space then we let  $\mathbb{S}(X)$  and  $\mathbb{P}(X)$  denote the sets of subprobability measures and probability measures on  $X$  respectively, and remark that each is also a standard Borel space [19]. We will also assume that the reader is familiar with the theory of integration. If  $\mu$  is a finite measure and  $f$  is an integrable function both defined on  $(X, \mathcal{B}_X)$  then we denote the integral of  $f$  with respect to  $\mu$  by  $\mu(f)$ .

Working at the level of standard Borel spaces allows us to use the rich structure of Polish metric spaces without necessarily having to fix a metric beforehand. For example, when examining probability measures on such spaces, we can sometimes restrict to compact metric spaces, which in turn provide finite substructure for estimation schemes or over which convergence of certain functions can be made to be uniform. The following can be found in [20] and [21] and will be used to establish Theorem 9 in Section 3.

#### Definition 2. Let $\mathcal{P}$ be a family of Borel probability measures on a metric space $(X, d)$ .

1.  $\mathcal{P}$  is said to be uniformly tight iff for every  $\epsilon > 0$  there exists a compact subset  $K$  of  $X$  such that  $P(X \setminus K) < \epsilon$  for every  $P \in \mathcal{P}$ .

2.  $\mathcal{P}$  is said to be relatively compact iff every sequence of elements in  $\mathcal{P}$  contains a weakly convergent subsequence.

**Theorem 1 (Prohorov’s Theorem).** *Suppose  $(X, d)$  is a Polish metric space. Let  $\mathcal{P} \subseteq \mathbb{P}(X)$ . Then  $\mathcal{P}$  is relatively compact if and only if it is uniformly tight.*

**Theorem 2 (Dini’s Theorem).** *Suppose  $(K, \tau)$  is a compact topological space,  $(f_n)_{n \in \mathbb{N}}$  is a sequence of continuous real-valued functions on  $K$ , monotonically decreasing and converging pointwise to a continuous function  $f$ . Then  $(f_n)_{n \in \mathbb{N}}$  converges to  $f$  uniformly on  $K$ .*

## 2.2 Stochastic Kernels and Markov Processes

**Definition 3.** *Let  $(X, \mathcal{B}_X)$  and  $(Y, \mathcal{B}_Y)$  be standard Borel spaces. A sub-Markov kernel<sup>4</sup> is a Borel measurable map from  $(X, \mathcal{B}_X)$  to  $(\mathbb{S}(Y), \mathcal{B}_{\mathbb{S}(Y)})$ . A Markov kernel is a Borel measurable map from  $(X, \mathcal{B}_X)$  to  $(\mathbb{P}(Y), \mathcal{B}_{\mathbb{P}(Y)})$ .*

Equivalently,  $K$  is a sub-Markov (Markov) kernel from  $X$  to  $Y$  iff

1.  $K(x)$  is a sub-probability (probability) measure on  $(Y, \mathcal{B}_Y)$  for each  $x \in X$ , and
2.  $x \mapsto K(x)(B)$  is a measurable map for each  $B \in \mathcal{B}_Y$ .

We will simply write “ $K$  is a sub-Markov (Markov) kernel on  $X$ ” when it is implicitly assumed that  $Y = X$ . Such stochastic kernels play the role of a transition relation in stochastic transition systems with continuous state spaces. The two Markov processes that we examine in detail are the labelled Markov process and the Markov decision process.

**Definition 4.** *A labelled Markov process (LMP) is a tuple  $(S, \mathcal{B}_S, A, \{K_a : a \in A\})$ , where  $(S, \mathcal{B}_S)$  is a standard Borel space,  $A$  is a finite set of labels, and for  $a \in A$ ,  $K_a$  is a sub-Markov kernel on  $S$ .*

**Definition 5.** *A Markov decision process (MDP) is a tuple  $(S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$ , where  $(S, \mathcal{B}_S)$  is a standard Borel space,  $A$  is a finite set of actions,  $r : A \times S \rightarrow \mathbb{R}$  is a bounded measurable reward function, and for  $a \in A$ ,  $P_a$  is a Markov kernel on  $S$ .*

For each  $a \in A$  we will denote by  $r_a : S \rightarrow \mathbb{R}$  the function defined by  $r_a(s) = r(a, s)$ .

*Remark 1.* In [9, 22, 23, 10], Desharnais, Panangaden, et al. consider LMPs in which the state spaces are analytic sets; this is largely because the quotient of a Polish space may fail to be Polish but is always guaranteed to be analytic. We will not consider analytic sets in this work, but the interested reader should keep this in mind.

## 2.3 Bisimulation

We present bisimilarity for LMPs and MDPs as outlined in [10] and [4] and note that the latter amounts to little more than a mild extension through the addition of rewards to the definition of the former.

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<sup>4</sup> This is also known as a stochastic relation, a stochastic transition kernel, or simply a stochastic kernel.

**Definition 6.** Given a relation  $R$  on a set  $S$ , a subset  $X$  of  $S$  is said to be  $R$ -closed if and only if the collection of all those elements of  $S$  that it is related to by  $R$ ,  $R(X) = \{s' \in S \mid \exists s \in X, sRs'\}$ , is itself contained in  $X$ .

**Definition 7.** Given a relation  $R$  on a measurable space  $(S, \Sigma)$ , we write  $\Sigma(R)$  for the set of those  $\Sigma$ -measurable sets that are also  $R$ -closed,  $\{X \in \Sigma \mid R(X) \subseteq X\}$ .

When  $R$  is an equivalence relation then to say that a set  $X$  is  $R$ -closed is equivalent to saying that  $X$  is a union of  $R$ -equivalence classes. In this case  $\Sigma(R)$  consists of those measurable sets that can be partitioned into  $R$ -equivalence classes.

**Definition 8.** Let  $(S, \mathcal{B}_S, A, \{K_a : a \in A\})$  be an LMP. An equivalence relation  $R$  on  $S$  is a bisimulation relation if and only if it satisfies

$$sRs' \Leftrightarrow \text{for every } a \in A \text{ and for every } X \in \Sigma(R), K_a(s)(X) = K_a(s')(X).$$

Bisimilarity is the largest of the bisimulation relations.

**Definition 9.** Let  $(S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP. An equivalence relation  $R$  on  $S$  is a bisimulation relation if and only if it satisfies

$$sRs' \Leftrightarrow \text{for every } a \in A, r_a(s) = r_a(s') \text{ and for every } X \in \Sigma(R), P_a(s)(X) = P_a(s')(X).$$

Bisimilarity is the largest of the bisimulation relations.

It turns out that bisimulation for LMPs and MDPs can be equivalently cast as the maximal fixed-point of a monotone functional on a complete lattice. We present this here only in the context of MDPs; the statement for LMPs is analogous.

**Theorem 3.** Let  $(S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP,  $\tau$  a Polish topology on  $S$  generating  $\mathcal{B}_S$  and such that for each  $a$  in  $A$ ,  $r_a$  and  $P_a$  are continuous with respect to  $\tau$ ,  $\mathbb{P}(S)$  being endowed with the topology of weak convergence induced by  $\tau$ . Assume that the image of  $r$  is contained in  $[0, 1]$ . Define  $\mathcal{F} : \mathfrak{Equ} \rightarrow \mathfrak{Equ}$  by

$$s\mathcal{F}(R)s' \Leftrightarrow \forall a \in A, r_a(s) = r_a(s') \text{ and } \forall X \in \Sigma(R), P_a(s)(X) = P_a(s')(X),$$

where  $\mathfrak{Equ}$  is the set of equivalence relations on  $S$  equipped with subset ordering. Then the greatest fixed point of  $\mathcal{F}$  is bisimilarity.

Lastly, we remark that bisimulation for LMPs has a logical characterization, and in turn, a characterization in terms of a real-valued modal logic. We omit the details for lack of space, but return to the latter idea in subsequent sections.

## 2.4 Probability Metrics

Metriizing bisimilarity for Markov processes essentially involves assigning a distance to their Markov kernels via a suitable probability metric. Gibbs and Su [24] survey a variety of such metrics. LMP bisimilarity was initially defined in terms of an integral probability metric in [10], and later recast in terms of the Kantorovich metric in [2]. In order to present the Kantorovich metric, we first recall the definition of lower semicontinuity.

**Definition 10.** Let  $(X, \tau)$  be a topological space and let  $f : X \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ . Then  $f$  is lower semicontinuous if for each half-open interval of the form  $(r, \infty)$ , the preimage  $f^{-1}(r, \infty) \in \tau$ .

## The Kantorovich Metric

**Definition 11.** Let  $S$  be a Polish space,  $h$  a bounded pseudometric on  $S$  that is lower semicontinuous on  $S \times S$  with respect to the product topology, and  $Lip(h)$  be the set of all bounded functions  $f : S \rightarrow \mathbb{R}$  that are measurable with respect to the Borel  $\sigma$ -algebra on  $S$  and that satisfy the Lipschitz condition  $f(x) - f(y) \leq h(x, y)$  for every  $x, y \in S$ . Let  $P, Q \in \mathbb{P}(S)$ . Then the Kantorovich metric  $\mathfrak{K}(h)$  is defined by

$$\mathfrak{K}(h)(P, Q) = \sup_{f \in Lip(h)} (P(f) - Q(f)).$$

**Lemma 1.** Let  $S, h, P,$  and  $Q$  be as in Definition 11. Let  $\Lambda(P, Q)$  consist of all measures on the product space  $S \times S$  with marginals  $P$  and  $Q$ , i.e.,

$$\Lambda(P, Q) = \{\lambda \in \mathbb{P}(S \times S) : \lambda(E \times S) = P(E) \text{ and } \lambda(S \times E) = Q(E) \text{ for all } E \in \mathcal{B}_S\}. \quad (1)$$

Then the Kantorovich metric  $\mathfrak{K}(h)$  satisfies the inequality:

$$\sup_{f \in Lip(h, C^b(S))} (P(f) - Q(f)) \leq \mathfrak{K}(h)(P, Q) \leq \inf_{\lambda \in \Lambda(P, Q)} \lambda(h) \quad (2)$$

where  $Lip(h, C^b(S))$  denotes functions on  $S$  that are continuous and bounded, 1-Lipschitz with respect to  $h$ , and have range  $[0, \|h\|]$ .

Note that  $h$  need not generate the topology on  $S$ , and so Lipschitz continuity with respect to  $h$  does not immediately imply continuity on  $S$ .

The leftmost and rightmost terms in inequality 2 are examples of infinite linear programs in duality. It is a highly nontrivial result that there is no duality gap in this case (see for example Theorem 1.3 and the proof of Theorem 1.14 in [25]).

**Theorem 4 (Kantorovich-Rubinstein Duality Theorem).** Assume the conditions of Definition 11 and Lemma 1. Then there is no duality gap in equation 2, that is,

$$\mathfrak{K}(h)(P, Q) = \sup_{f \in Lip(h, C^b(S))} (P(f) - Q(f)) = \inf_{\lambda \in \Lambda(P, Q)} \lambda(h) \quad (3)$$

Note that for any point masses  $\delta_x, \delta_y$ , we have  $\mathfrak{K}(h)(\delta_x, \delta_y) = h(x, y)$  since  $\delta_{(x,y)}$  is the only measure with marginals  $\delta_x$  and  $\delta_y$  on the right-hand side of Equation 3. As a result, we obtain that any bounded lower semicontinuous pseudometric  $h$  can be expressed as  $h(x, y) = \sup_{f \in \mathcal{F}} (f(x) - f(y))$  for some family of continuous functions  $\mathcal{F}$ .

**Integral Probability Metrics** The intuition behind the Kantorovich metric is that the quantitative difference between two probability measures can be measured in terms of the maximal difference between the expected values with respect to the two measures, of a class of test functions - in this case, the class of Lipschitz functions. For an arbitrary class of test functions, the induced metric is known as the integral probability metric generated by that class. All definitions and results of this subsection are taken from [5].

**Definition 12.** Let  $\mathcal{F}$  be a subset of bounded measurable real-valued functions on a Polish metric space  $(S, d)$ . Then the integral probability metric associated with  $\mathcal{F}$  is the probability metric  $IPM(\mathcal{F})$  on  $\mathbb{P}(S)$  defined by

$$IPM(\mathcal{F})(P, Q) = \sup_{f \in \mathcal{F}} |P(f) - Q(f)|$$

for probability measures  $P$  and  $Q$ .

For convenience, we will simply denote  $IPM(\mathcal{F})$  by  $\mathcal{F}$ . Remark that in general  $\mathcal{F}$  is allowed to take on infinite values, though we will work with bounded sets of functions to avoid this. Additionally, we remark that this probability metric in turn induces a pseudometric on  $S$  via

$$\mathcal{F}(x, y) := \mathcal{F}(\delta_x, \delta_y).$$

Thus, as an abuse of notation we will use  $\mathcal{F}$  to refer to a family of functions, the associated probability metric, and the induced pseudometric, with the intended meaning clear from the context.

**Definition 13.** Let  $\mathcal{F}$  be a subset of bounded measurable real-valued functions on a Polish metric space  $(S, d)$ . The maximal generator of the integral probability metric associated to  $\mathcal{F}$  is the set  $\mathcal{R}_{\mathcal{F}}$  of all bounded measurable real-valued functions on  $(S, d)$ , each of which satisfies the following:  $g \in \mathcal{R}_{\mathcal{F}}$  if and only if

$$|P(g) - Q(g)| \leq \mathcal{F}(P, Q)$$

for every  $P$  and  $Q$  in  $\mathbb{P}(S)$ .

It follows that  $\mathcal{R}_{\mathcal{F}}$  is the largest such family, and that  $\mathcal{R}_{\mathcal{F}}(P, Q) = \mathcal{F}(P, Q)$ .

The following is Theorem 3.3 of [5].

**Theorem 5.** Let  $\mathcal{F}$  be an arbitrary generator of  $\mathcal{R}_{\mathcal{F}}$ . Then

1.  $\mathcal{R}_{\mathcal{F}}$  contains the convex hull of  $\mathcal{F}$ ;
2.  $f \in \mathcal{R}_{\mathcal{F}}$  implies  $\alpha f + \beta \in \mathcal{R}_{\mathcal{F}}$  for all  $\alpha \in [0, 1]$  and  $\beta \in \mathbb{R}$ ;
3. If the sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathcal{R}_{\mathcal{F}}$  converges uniformly to  $f$ , then  $f \in \mathcal{R}_{\mathcal{F}}$ .

In particular, for a given  $\mathcal{F}$ ,  $\mathcal{R}_{\mathcal{F}}$  is closed under uniform convergence.

## 2.5 Bisimulation Metrics

The metric analogue of bisimulation for LMPs was initially cast in terms of a family of functional expressions, interpreted as a real-valued logic over the states of a given Markov process [9]. A slightly modified version was then shown to be equivalent to a bisimulation metric developed in the context of category theory in [2]. In [1], the authors in turn recast this latter metric fully using order-theoretic fixed-point theory for discrete systems. Finally, this method was generalized to develop a bisimulation metric for MDPs with continuous state spaces in [4].

We present here the logic of [2] and the fixed-point results of [4], as these are the results we will attempt to join in the subsequent sections.

**Definition 14.** For each  $c \in (0, 1]$ ,  $\mathcal{F}_c$  represents the family of functional expressions generated by the following grammar.

$$f ::= \mathbf{1} \mid \mathbf{1} - f \mid \langle a \rangle f \mid \max(f, f) \mid f \ominus q \tag{4}$$

where  $q \in \mathbb{Q} \cap [0, 1]$  and  $a$  belongs to a fixed set of labels  $A$ .

Let  $\mathcal{M} = (S, \mathcal{B}_S, A, \{K_a : a \in A\})$  be an LMP. The interpretation of  $f \in \mathcal{F}_c$ ,  $f_{\mathcal{M}} : S \rightarrow [0, 1]$ , is defined inductively. Let  $s \in S$ . Then

$$\begin{aligned} \mathbf{1}_{\mathcal{M}}(s) &= 1 \\ (\mathbf{1} - f)_{\mathcal{M}}(s) &= 1 - f_{\mathcal{M}}(s) \\ \langle a \rangle f_{\mathcal{M}}(s) &= c \cdot K_a(s)(f_{\mathcal{M}}) \\ \max(f_1, f_2)_{\mathcal{M}}(s) &= \max(f_1_{\mathcal{M}}(s), f_2_{\mathcal{M}}(s)) \\ (f \ominus q)_{\mathcal{M}}(s) &= \max(f_{\mathcal{M}}(s) - q, 0), \end{aligned}$$

Henceforth, we shall omit the subscript  $\mathcal{M}$  and use  $f$  to refer both to an expression and its interpretation, with the difference clear from the context.

*Remark 2.* We may also add the expressions  $\min(f, f)$  and  $f \oplus q$  as shorthand for the expressions  $1 - \max(1 - f, 1 - f)$  and  $1 - ((1 - f) \ominus q)$ . The operations  $\ominus$  and  $\oplus$  denoted truncated subtraction in the unit interval and truncated addition in the unit interval, respectively.

The relevance of such a formulation arises via a behavioural pseudometric.

The following is Theorem 3 of [2] and Theorem 8.2 of [18].

**Theorem 6.** *Let  $\mathcal{M} = (S, \mathcal{B}_S, A, \{K_a : a \in A\})$  be an LMP and for  $c \in (0, 1]$ , let  $\mathcal{F}_c$  be the family of functional expressions defined in Definition 14. Define the map  $d_c$  on  $S \times S$  as follows:*

$$d_c(x, y) = \sup_{\mathcal{F}_c} |f(x) - f(y)|. \quad (5)$$

Then  $d_c$  is a pseudometric on  $S$  whose kernel is bisimilarity.

As previously mentioned, the metric  $d_c$  can be formulated in terms of fixed-point theory, and indeed this construction has been carried over to MDPs, with the minor addition of taking into account reward differences. The following is Theorem 3.12 of [4].

**Theorem 7.** *Let  $M = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP and let  $\tau$  be a Polish topology on  $S$  that generates  $\mathcal{B}_S$ . Assume that the image of  $r$  is contained in  $[0, 1]$ , and that for each  $a$  in  $A$ ,  $r_a$  and  $P_a$  are continuous,  $\mathbb{P}(S)$  endowed with the weak topology induced by  $\tau$ . Let  $c \in (0, 1)$  be a discount factor, and  $\mathbf{lsc}_m$  be the set of bounded pseudometrics on  $S$  that are lower semicontinuous on  $S \times S$  endowed with the product topology induced by  $\tau$ . Define  $F_c : \mathbf{lsc}_m \rightarrow \mathbf{lsc}_m$  by*

$$F_c(h)(s, s') = \max_{a \in A} ((1 - c)|r_a(s) - r_a(s')| + c \cdot \mathfrak{K}(h)(P_a(s), P_a(s')))$$

where  $\mathfrak{K}(h)$  is the Kantorovich metric induced by  $h \in \mathbf{lsc}_m$ . Then

1.  $F_c$  has a unique fixed point  $\rho_c^* : S \times S \rightarrow [0, 1]$ ,
2. The kernel of  $\rho_c^*$  is bisimilarity,
3. for any  $h_0 \in \mathbf{lsc}_m$ ,  $\lim_{n \rightarrow \infty} F_c^n(h_0) = \rho_c^*$ ,
4.  $\rho_c^*$  is continuous on  $S \times S$ ,
5.  $\rho_c^*$  is continuous in  $r$  and  $P$ , and
6. If MDP  $M' = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, k \cdot r)$  for some  $k \in [0, 1]$  then  $\rho_{c, M'}^* = k \cdot \rho_{c, M}^*$ .

Whereas the interest in finding small bisimilar systems for LMPs lies in being able to test properties of a system specified in a given logic, the interest in finding small bisimilar systems for MDPs concerns finding optimal planning strategies in terms of value functions. Given a discount factor  $\gamma \in [0, 1)$ , the optimal value function is the unique solution to the following Bellman optimality fixed-point equation.

$$v^*(s) = \max_{a \in A} (r_a(s) + \gamma P_a(s)(v^*)) \text{ for each } s \in S.$$

In general, such a  $v^*$  need not exist. Even if it does, there may not be a well-behaved, that is to say measurable, policy that is captured by it. Fortunately, there are several mild restrictions under which this is not the case. According to Theorem 8.3.6 and its preceding remarks in [26], if the state space is Polish and the reward function is uniformly bounded then there exists a unique solution  $v^*$  to the Bellman optimality equation and there exists a measurable optimal policy for it as well.

The following is Theorem 3.20 in [4].

**Theorem 8.** *Let  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP and let  $\tau$  be a Polish topology on  $S$  that generates  $\mathcal{B}_S$ . Assume that the image of  $r$  is contained in  $[0, 1]$ , and that for each  $a$  in  $A$ ,  $r_a$  and  $P_a$  are continuous,  $\mathbb{P}(S)$  endowed with the weak topology induced by  $\tau$ . Let  $c \in (0, 1)$  be a discount factor. Let  $v_\gamma^*$  be the optimal value function for the expected total discounted reward associated with  $\mathcal{M}$  and discount factor  $\gamma \in [0, 1)$ . Suppose  $\gamma \leq c$ . Then  $v_\gamma^*$  is Lipschitz continuous with respect to  $\rho_c^*$  with Lipschitz constant  $\frac{1}{1-c}$ , i.e.,  $|v_\gamma^*(x) - v_\gamma^*(y)| \leq (1-c)^{-1} \rho_c^*(x, y)$ .*

### 3 Bisimulation Metrics for MDPs Revisited

The goal of this section is two-fold. First, we establish a family of functional expressions as in Definition 14 that captures bisimulation for MDPs as defined in Theorem 7. This amounts to little more than Proposition 2 of [2] but using the terminology of generating classes for integral probability metrics. Second, we generalize the applicability of these results for MDPs by removing the continuity constraints in Theorem 7.

#### 3.1 When is the Integral Probability Metric the Kantorovich Metric?

In this section we will show that under some very mild conditions, the maximal generator of a family of functional expressions is in fact the class of Lipschitz functions with respect to the distance induced by that family. In this case, the integral probability metric and the Kantorovich metric induced by the family coincide.

The following result is Lemma 4.6 of [1], itself adapted from Proposition 2 of [2], presented almost verbatim. The imposed Lipschitz condition makes measurability concerns almost an afterthought. What really matters here is the reframing of the result in terms of the integral probability metric and its maximal generator. Doing so will allow us to examine simpler grammars for bisimulation, as well as ways of approximating these.

**Theorem 9.** *Suppose  $(S, d)$  is a Polish metric space and  $\mathcal{F}$  is a family of real-valued functions on  $S$  that take values in the unit interval and are 1-Lipschitz continuous with respect to  $d$ . Suppose further that  $\mathcal{F}$  contains the constant zero function and is closed under truncated addition with rationals in the unit interval, subtraction from the constant function 1, and taking the pointwise maximum of two functions. Let  $\mathcal{R}_\mathcal{F}$  be the maximal generator of  $\mathcal{F}$  and  $Lip(\mathcal{F})$  be the set of real-valued measurable functions on  $S$  that are 1-Lipschitz with respect to the metric induced by  $\mathcal{F}$ . Then  $\mathcal{R}_\mathcal{F} = Lip(\mathcal{F}) \subseteq \mathbb{C}^b(S)$ .*

*Proof.* Firstly note that since by assumption all members of the family  $\mathcal{F}$  are 1-Lipschitz continuous with respect to  $d$ , the induced pseudometric  $\mathcal{F} \leq d$ . Thus,  $Lip(\mathcal{F}) \subseteq Lip(d) \subseteq \mathbb{C}^b(S)$ . From the definition of  $\mathcal{R}_\mathcal{F}$  applied to Dirac measures, it immediately follows that each of its members is 1-Lipschitz with respect to the pseudometric induced by  $\mathcal{F}$ . Thus,  $\mathcal{R}_\mathcal{F} \subseteq Lip(\mathcal{F})$ . In particular, every member of the maximal generator belongs to  $\mathbb{C}^b(S)$ .

The reverse inclusion  $Lip(\mathcal{F}) \subseteq \mathcal{R}_\mathcal{F}$  is somewhat more complicated to establish. By Theorem 5, we have that  $\mathcal{R}_\mathcal{F}$  is closed with respect to uniform convergence, and thus is also generated by  $\overline{\mathcal{F}}$ , the closure of  $\mathcal{F}$  with respect to uniform convergence. In fact, we will show that  $\mathcal{F}$  is dense in  $Lip(\mathcal{F})$  in the metric of uniform convergence; for then it follows that  $Lip(\mathcal{F}) = \overline{\mathcal{F}} \subseteq \mathcal{R}_\mathcal{F}$ . We do so in two steps. First we establish the result in the case where  $(S, d)$  is a compact metric space, as this allows us to replace pointwise convergence by uniform convergence at a certain point in the

proof. Finally, we extend this result to the general case of a Polish metric space by approximating it from within by a suitable compact subset.

Assume that  $(S, d)$  is a compact metric space. It is easily seen that  $\overline{\mathcal{F}}$  contains the constant zero function and remains closed under truncated addition with *all* constants in the unit interval, subtraction from 1, and taking maxima; in fact, it now follows that  $\overline{\mathcal{F}}$  is closed under countable suprema. To see this, suppose  $(f_n)_{n \in \mathbb{N}}$  is a sequence in  $\overline{\mathcal{F}}$ . Since  $\overline{\mathcal{F}}$  is uniformly bounded by 1 it follows that  $f = \sup_{n \in \mathbb{N}} f_n$  exists and moreover it is continuous, as each  $f_n$  is 1-Lipschitz continuous with respect to  $d$ . Define  $(h_n)_{n \in \mathbb{N}}$  in  $\overline{\mathcal{F}}$  by  $h_n = \max_{1 \leq i \leq n} f_i$ . Then  $(h_n)_{n \in \mathbb{N}}$  is monotonically increasing and converges pointwise to  $f$ . By Theorem 2,  $(h_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$ , and hence  $f$  belongs to  $\overline{\mathcal{F}}$ . It now also follows that  $\overline{\mathcal{F}}$  is closed under truncated subtraction with constants in the unit interval, taking minima, and taking infima.

Let  $g \in Lip(\mathcal{F})$ . Without loss of generality, we assume the image of  $g$  belongs to  $[0, 1]$ ; for the Lipschitz property with respect to  $\mathcal{F}$  implies that  $\sup g - \inf g \leq 1$  and we may replace  $g$  by  $g' := g - \inf g$ . It follows that if  $g'$  belongs to  $\mathcal{R}_{\mathcal{F}}$  then so does  $g = g' + \inf g$ .

Let  $\epsilon > 0$ . Then there exists  $f_{xy\epsilon} \in \mathcal{F}$  such that

$$g(x) - g(y) \leq \mathcal{F}(x, y) \leq f_{xy\epsilon}(x) - f_{xy\epsilon}(y) + \epsilon \quad (6)$$

Define  $w_{xy\epsilon} \in \overline{\mathcal{F}}$  as follows:

$$w_{xy\epsilon}(z) = \begin{cases} f_{xy\epsilon}(z) & \text{if } f_{xy\epsilon}(x) = g(x) \\ f_{xy\epsilon}(z) \ominus_{[0,1]} (f_{xy\epsilon}(x) - g(x)) & \text{if } f_{xy\epsilon}(x) > g(x) \\ f_{xy\epsilon}(z) \oplus_{[0,1]} (g(x) - f_{xy\epsilon}(x)) & \text{if } f_{xy\epsilon}(x) < g(x) \end{cases} \quad (7)$$

Then  $w_{xy\epsilon}(x) = g(x)$  and  $w_{xy\epsilon}(y) \leq g(y) + \epsilon$ .

Let  $(u_n)_{n \in \mathbb{N}}$  be a dense sequence in  $(S, d)$ . Define  $(\phi_{nm\epsilon})_{n, m \in \mathbb{N}} \subseteq \overline{\mathcal{F}}$  by  $\phi_{nm\epsilon} = w_{u_n u_m \epsilon}$  and define  $(\phi_{n\epsilon})_{n \in \mathbb{N}}$  by  $\phi_{n\epsilon} = \inf_{m \in \mathbb{N}} \phi_{nm\epsilon}$ . It follows that  $(\phi_{n\epsilon})_{n \in \mathbb{N}} \subseteq \overline{\mathcal{F}}$ . Moreover,

$$\phi_{n\epsilon}(u_n) = g(u_n) \leq g(u_n) + \epsilon \text{ and for each } m \neq n, \phi_{n\epsilon}(u_m) \leq g(u_m) + \epsilon.$$

Define  $\psi_\epsilon = \sup_{n \in \mathbb{N}} \phi_{n\epsilon} \in \overline{\mathcal{F}}$ . Then for any  $n \in \mathbb{N}$ ,

$$g(u_n) \leq \psi_\epsilon(u_n) \leq g(u_n) + \epsilon. \quad (8)$$

Let  $x \in S$ . Then as the inequalities in line 8 hold for any subsequence of  $(u_n)_{n \in \mathbb{N}}$  converging to  $x$ , and as both  $g$  and  $\psi_\epsilon$  are continuous, it follows by taking limits that for any  $x \in S$ ,

$$g(x) \leq \psi_\epsilon(x) \leq g(x) + \epsilon, \text{ or equivalently } \|\psi_\epsilon - g\| < \epsilon.$$

Define the sequence  $(g_n)_{n \in \mathbb{N}}$  in  $\overline{\mathcal{F}}$  by  $g_n = \psi_{\frac{1}{n}}$ . Then  $(g_n)_{n \in \mathbb{N}}$  converges uniformly to  $g$ . Therefore,  $g$  belongs to  $\overline{\mathcal{F}} \subseteq \mathcal{R}_{\mathcal{F}}$ , i.e.  $Lip(\mathcal{F}) \subseteq \mathcal{R}_{\mathcal{F}}$ .

Now suppose  $(S, d)$  is a general Polish metric space. Let  $P, Q \in \mathbb{P}(S)$ . Then  $\mathcal{P} = \{P, Q\}$  is finite, hence relatively compact. By Theorem 1,  $\mathcal{P}$  is uniformly tight. Let  $0 < \epsilon < \frac{1}{2}$ . Then there exists a compact subset  $K$  of  $S$  such that  $P(S \setminus K) < \epsilon$  and  $Q(S \setminus K) < \epsilon$ .

Let  $\mathcal{G}$  denote the functions of  $\mathcal{F}$  restricted to  $K$ ; for  $f \in \mathcal{F}$ , we will write  $f_K \in \mathcal{G}$ . Then as  $\mathcal{G}$  still contains the constant zero function, and is closed under the same operations as  $\mathcal{F}$ , and as  $(K, d)$  is a compact metric space, we have  $\mathcal{R}_{\mathcal{G}} = Lip(\mathcal{G})$ . Let  $g \in Lip(\mathcal{F})$ ; as before, we assume

without loss of generality that the image of  $g$  is contained in  $[0, 1]$ . Moreover, let  $g_K$  be  $g$  restricted to  $K$  and remark that  $g_K \in Lip(\mathcal{G})$ . Next we define  $P_K, Q_K \in \mathbb{P}(K)$  by

$$P_K(E) = \frac{P(E \cap K)}{P(K)} \text{ and } Q_K(E) = \frac{Q(E \cap K)}{Q(K)}.$$

Remark that  $P(K) > 1 - \epsilon > \frac{1}{2}$ , and similarly for  $Q(K)$ , so that each is well-defined. Then as  $g_K \in \mathcal{R}_g$ ,

$$|P_K(g_K) - Q_K(g_K)| \leq \mathcal{G}(P_K, Q_K).$$

Next for any 1-bounded measurable function  $u$  on  $S$  and its restriction  $u_K$  to  $K$ , we have

$$\begin{aligned} |P(u) - P_K(u_K)| &= |P(u \cdot \delta_K) + P(u \cdot \delta_{S \setminus K}) - P_K(u_K)| = |(P(u \cdot \delta_K) - P_K(u_K)) + P(u \cdot \delta_{S \setminus K})| \\ &\leq |1 - \frac{1}{P(K)}| \cdot P(u \cdot \delta_K) + 1 \cdot P(S \setminus K) \leq \frac{1}{1 - \epsilon} - 1 + \epsilon \leq \frac{\epsilon(2 - \epsilon)}{1 - \epsilon} \leq 3\epsilon, \end{aligned}$$

where  $\delta_K$  is the indicator function on  $K$ . Similarly  $|Q(u) - Q_K(u_K)| \leq 3\epsilon$ . Finally,

$$\begin{aligned} |P(g) - Q(g)| &\leq |P(g) - P_K(g_K)| + |P_K(g_K) - Q_K(g_K)| + |Q_K(g_K) - Q(g)| \\ &\leq 3\epsilon + \mathcal{G}(P_K, Q_K) + 3\epsilon \leq 6\epsilon + \sup_{f \in \mathcal{F}} |P_K(f_K) - Q_K(f_K)| \\ &\leq 6\epsilon + \sup_{f \in \mathcal{F}} (|P_K(f_K) - P(f)| + |P(f) - Q(f)| + |Q(f) - Q_K(f_K)|) \\ &\leq 12\epsilon + \sup_{f \in \mathcal{F}} (|P(f) - Q(f)|) \leq 12\epsilon + \mathcal{F}(P, Q). \end{aligned}$$

As  $\epsilon$  is arbitrary, it follows that  $|P(g) - Q(g)| \leq \mathcal{F}(P, Q)$  and  $g \in \mathcal{R}_{\mathcal{F}}$ . □

**Corollary 1.** *Suppose  $(S, d)$  is a Polish metric space and  $\mathcal{F}$  is a family of real-valued functions on  $S$  that take values in the unit interval and are 1-Lipschitz continuous with respect to  $d$ . Suppose further that  $\mathcal{F}$  contains the constant zero function and is closed under truncated addition with rationals in the unit interval, subtraction from the constant function 1, and taking the pointwise maximum of two functions. Then the integral probability metric associated to  $\mathcal{F}$  is the Kantorovich metric of the pseudometric induced by  $\mathcal{F}$ , i.e.  $\mathcal{F}(P, Q) = \mathfrak{K}(\mathcal{F})(P, Q)$  for any  $P, Q \in \mathbb{P}(S)$ .*

### 3.2 A Family of Functional Expressions for MDP Bisimulation

We now define a family of functional expressions as in Definition 14 that when evaluated on a given MDP, capture bisimilarity.

**Definition 15.** *For each  $c \in (0, 1]$ ,  $\mathcal{F}_c$  represents the family of functional expressions generated by the following grammar.*

$$f ::= \mathbf{0} \mid \mathbf{1} - f \mid \langle a \rangle f \mid \max(f, f) \mid f \oplus q \tag{9}$$

where  $q \in \mathbb{Q} \cap [0, 1]$  and  $a$  belongs to a fixed set of labels  $A$ .

Let  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP. The interpretation of  $f \in \mathcal{F}_c$ ,  $f_{\mathcal{M}} : S \rightarrow [0, 1]$ , is defined inductively. Let  $s \in S$ . Then

$$\begin{aligned} \mathbf{0}_{\mathcal{M}}(s) &= 0 \\ (\mathbf{1} - f)_{\mathcal{M}}(s) &= 1 - f_{\mathcal{M}}(s) \\ \langle a \rangle f_{\mathcal{M}}(s) &= r_a(s) + c \cdot P_a(s)(f_{\mathcal{M}}) \\ \max(f_1, f_2)_{\mathcal{M}}(s) &= \max(f_1)_{\mathcal{M}}(s), (f_2)_{\mathcal{M}}(s) \\ (f \oplus q)_{\mathcal{M}}(s) &= \min(f_{\mathcal{M}}(s) + q, 1). \end{aligned}$$

As before, we shall omit the subscript  $\mathcal{M}$  when it is clear from the context, and remark that the family also contains the expressions  $\min(f, f)$  and  $f \ominus q$ .

We now show that the integral probability metric generated by  $\mathcal{F}_c$  agrees with the Kantorovich metric induced by the fixed-point bisimulation metric for MDPs. This is essentially the proof method in all of Section 4 of [1].

**Theorem 10.** *Suppose  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  is an MDP and let  $\tau$  be a Polish topology on  $S$  that generates  $\mathcal{B}_S$ . Assume that the image of  $r$  is contained in  $[0, 1]$ , and that for each  $a$  in  $A$ ,  $r_a$  and  $P_a$  are continuous,  $\mathbb{P}(S)$  endowed with the weak topology induced by  $\tau$ . Let  $c \in (0, 1)$  be a discount factor, and  $\mathcal{F}_c$  be the family of functional expressions defined in Definition 15. Let  $\mathcal{G}$  be a family of functional expressions such that  $\mathcal{F}_c \subseteq \mathcal{G} \subseteq \text{Lip}(\mathcal{F}_c)$ . Then the pseudometric induced by  $\mathcal{G}$  coincides with the fixed-point metric  $\rho_c^*$  given by Theorem 7.*

*Proof.* Let  $(S, d)$  be a Polish metric space such that  $d$  generates  $\tau$ . Since  $\rho_c^*$  is continuous, we can assume without loss of generality that  $\rho_c^* \leq d$ , as we can simply replace  $d$  by the equivalent metric  $d + \rho_c^*$ . By structural induction,  $\mathcal{F}_c \subseteq \rho_c^* \leq d$ , and the range of each member of  $\mathcal{F}_c$  is  $[0, 1]$ . Therefore by Corollary 1, the integral probability metric and the Kantorovich metric induced by  $\mathcal{F}_c$  agree.

Notice that since  $\mathcal{F}_c$  is closed under subtraction from the constant function 1, we have that for any  $P, Q \in \mathbb{P}(S)$

$$\begin{aligned} \mathcal{F}_c(P, Q) &= \sup_{f \in \mathcal{F}_c} |P(f) - Q(f)| = \sup_{f \in \mathcal{F}_c} \max(P(f) - Q(f), Q(f) - P(f)) \\ &= \max(\sup_{f \in \mathcal{F}_c} P(f) - Q(f), \sup_{f \in \mathcal{F}_c} Q(f) - P(f)) \\ &= \max(\sup_{f \in \mathcal{F}_c} P(f) - Q(f), \sup_{f \in \mathcal{F}_c} P(1 - f) - Q(1 - f)) \\ &= \max(\sup_{f \in \mathcal{F}_c} P(f) - Q(f), \sup_{f \in \mathcal{F}_c} P(f) - Q(f)) \\ &= \sup_{f \in \mathcal{F}_c} P(f) - Q(f) \end{aligned}$$

which is not necessarily the case otherwise. A simple structural induction next shows that

$$\mathcal{F}_c(x, y) = \sup_{a \in A, f \in \mathcal{F}_c} |\langle a \rangle f(x) - \langle a \rangle f(y)|.$$

Therefore,

$$\begin{aligned}
\mathcal{F}_c(x, y) &= \sup_{a \in A, f \in \mathcal{F}_c} \max(\langle a \rangle f(x) - \langle a \rangle f(y), \langle a \rangle f(y) - \langle a \rangle f(x)) \\
&= \sup_{a \in A, f \in \mathcal{F}_c} \max\left( (1-c)(r_a(x) - r_a(y)) + c(P_a(x)(f) - P_a(y)(f)), \right. \\
&\quad \left. (1-c)(r_a(y) - r_a(x)) + c(P_a(y)(f) - P_a(x)(f)) \right) \\
&= \max_{a \in A} \max\left( (1-c)(r_a(x) - r_a(y)) + c \sup_{f \in \mathcal{F}_c} (P_a(x)(f) - P_a(y)(f)), \right. \\
&\quad \left. (1-c)(r_a(y) - r_a(x)) + c \sup_{f \in \mathcal{F}_c} (P_a(y)(f) - P_a(x)(f)) \right) \\
&= \max_{a \in A} \max\left( (1-c)(r_a(x) - r_a(y)) + c \cdot \mathcal{F}_c(P_a(x), P_a(y)), \right. \\
&\quad \left. (1-c)(r_a(y) - r_a(x)) + c \cdot \mathcal{F}_c(P_a(x), P_a(y)) \right) \\
&= \max_{a \in A} (1-c) \max\left( (r_a(x) - r_a(y)), (r_a(y) - r_a(x)) \right) + c \cdot \mathcal{F}_c(P_a(x), P_a(y)) \\
&= \max_{a \in A} (1-c) |r_a(x) - r_a(y)| + c \cdot \mathcal{F}_c(P_a(x), P_a(y)) \\
&= \max_{a \in A} (1-c) |r_a(x) - r_a(y)| + c \cdot \mathfrak{R}(\mathcal{F}_c)(P_a(x), P_a(y)) \\
&= F_c(\mathcal{F}_c)(x, y).
\end{aligned}$$

The penultimate line follows from Corollary 1. Therefore,  $\mathcal{F}_c$  is a fixed-point of the functional  $F_c$  defined in Theorem 7. As the fixed-point is unique, it follows that  $\rho_c^* = \mathcal{F}_c$ . Finally, it follows from Theorem 9 and the definition of maximal generator that  $\mathcal{G} = \mathcal{F}_c = \rho_c^*$ .  $\square$

*Remark 3.* Theorem 10 provides another proof of Theorem 8. Consider the family  $\mathcal{G}$  with the expression for the Bellman operator  $\mathcal{B}_\gamma$  for the MDP  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, (1-c)r)$  and discount factor  $\gamma \leq c$  in  $[0, 1)$ . Since  $Lip(\mathcal{F}_c)$  is closed under  $\mathcal{B}_\gamma$  and the optimal value function scales with rewards, the result follows immediately. Otherwise, we obtain the result only for  $V_c^*$  since  $\mathcal{B}_c(f) = \max_{a \in A} \langle a \rangle f$ .

The usefulness of this theorem derives from our choice of  $\mathcal{G}$ . On the one hand, we might attempt to see what is the minimal family, if one exists, that captures bisimilarity. On the other hand, we might consider explicitly adding operators, like the Bellman operator, that could help an estimation scheme converge faster. We will explore this further in Section 4.

Practical application is still hindered by the continuity constraints on the rewards and Markov kernels, as many interesting problems model discontinuous phenomena. In the next section, we will work to remove these constraints.

### 3.3 The General Case: Continuity from Measurability

We conclude this section with a neat little result from descriptive set theory that was first pointed out to the authors by Ernst-Erich Doberkat at the 2012 Bellairs Workshop on Probabilistic Systems organized by Prakash. In the most interesting reinforcement learning applications, continuity of the reward process cannot be guaranteed. Amazingly, we may remove the explicit assumption of continuity in [4] and the result still holds! We seek to establish the following.

**Theorem 11.** *Let  $(X, \tau)$  be a Polish space and  $(\mathbb{P}(X), \tau_{\mathbb{P}(X)})$  be the space of probability measures on  $X$  equipped with the topology of weak convergence with respect to  $\tau$ . Let  $K : (X, \mathcal{B}_X) \rightarrow (\mathbb{P}(X), \mathcal{B}_{\mathbb{P}(X)})$  be a stochastic kernel. Then there exists a finer Polish topology  $\tau'$  on  $X$  such that  $\sigma(\tau') = \mathcal{B}_X$ ,  $\sigma(\tau'_{\mathbb{P}(X)}) = \mathcal{B}_{\mathbb{P}(X)}$ , and  $K : (X, \tau') \rightarrow (\mathbb{P}(X), \tau'_{\mathbb{P}(X)})$  is continuous.*

This result is a minor reworking of the following well-known measurability-to-continuity theorem, which is Corollary 3.2.6 in [27].

**Theorem 12.** *Suppose  $(X, \tau)$  is a Polish space,  $Y$  a separable metric space, and  $f : X \rightarrow Y$  a Borel map. Then there is a finer Polish topology  $\tau'$  on  $X$  generating the same Borel  $\sigma$ -algebra such that  $f : (X, \tau') \rightarrow Y$  is continuous.*

We will also make use of this characterization of Borel  $\sigma$ -algebra on the set of probability measures, which is Proposition 7.25 in [28].

**Proposition 1.** *Let  $X$  be a separable metrizable space and  $\mathbb{E}$  a collection of subsets of  $X$  which generates  $\mathcal{B}_X$  and is closed under finite intersections. Then  $\mathcal{B}_{P(X)}$  is the smallest  $\sigma$ -algebra with respect to which all functions of the form  $\Theta_E(p) = p(E)$ , for  $E \in \mathbb{E}$ , are measurable from  $P(X)$  to  $[0, 1]$ , i.e.,*

$$\mathcal{B}_{P(X)} = \sigma[\cup_{E \in \mathbb{E}} \Theta_E^{-1}(\mathcal{B}_R)].$$

For ease of exposition, we will divide the result into the following series of steps.

**Lemma 2.** *Let  $(X, \tau)$  and  $K$  be as in Theorem 11. Then there exists an increasing sequence  $(\tau_n)_{n \in \mathbb{N}}$  of Polish topologies on  $X$  finer than  $\tau$  such that  $\sigma(\tau_n) = \mathcal{B}_X$  and  $K : (X, \tau_{n+1}) \rightarrow (\mathbb{P}(X), (\tau_n)_{\mathbb{P}(X)})$  is continuous for all  $n \in \mathbb{N}$ .*

*Proof.* By Proposition 1 for any Polish topology  $\tau'$  generating  $\mathcal{B}_X$ ,  $\tau'_{\mathbb{P}(X)}$  generates  $\mathcal{B}_{\mathbb{P}(X)}$ . It is well known [29] that  $\tau'_{\mathbb{P}(X)}$  is also a Polish topology. Therefore,  $K : (X, \tau) \rightarrow (\mathbb{P}(X), \tau_{\mathbb{P}(X)})$  is a Borel map. By Theorem 12, there exists a finer Polish topology  $\tau_0$  such that  $\sigma(\tau_0) = \mathcal{B}_X$  and  $K : (X, \tau_0) \rightarrow (\mathbb{P}(X), \tau_{\mathbb{P}(X)})$  is continuous; but then  $K : (X, \tau_0) \rightarrow (\mathbb{P}(X), (\tau_0)_{\mathbb{P}(X)})$  is Borel. Repeating this argument, there exists a finer topology  $\tau_1$  on  $X$  such that  $\sigma(\tau_1) = \mathcal{B}_X$  and  $K : (X, \tau_1) \rightarrow (\mathbb{P}(X), (\tau_0)_{\mathbb{P}(X)})$  is continuous. The result now easily follows for all  $n \in \mathbb{N}$  by induction.  $\square$

**Lemma 3.** *Let  $(X, \tau)$ ,  $K$ , and  $(\tau_n)_{n \in \mathbb{N}}$  be as in Lemma 2. Then the least upper bound topology  $\tau_\infty = \vee_{n \in \mathbb{N}} \tau_n$  exists and is Polish,  $\sigma(\tau_\infty) = \mathcal{B}_X$ , and  $K : (X, \tau_\infty) \rightarrow (\mathbb{P}(X), (\tau_n)_{\mathbb{P}(X)})$  is continuous for all  $n \in \mathbb{N}$ .*

*Remark 4 ([27] Observation 2, pg. 93).* Let  $(\tau_n)_{n \in \mathbb{N}}$  be a sequence of Polish topologies on  $X$  such that for any two distinct elements  $x, y$  of  $X$ , there exist disjoint sets  $U, V \in \cap_{n \in \mathbb{N}} \tau_n$  such that  $x \in U$  and  $y \in V$ . Then the topology  $\tau_\infty$  generated by  $\cup_{n \in \mathbb{N}} \tau_n$  is Polish.

*Proof.* By Remark 4,  $\tau_\infty$  exists, is Polish, and is generated by  $\cup_{n \in \mathbb{N}} \tau_n$ . So  $\cup_{n \in \mathbb{N}} \tau_n$  is a subbasis for  $\tau_\infty$ . Let  $O \in \tau_\infty$ . Then  $O$  is an arbitrary union of finite intersections of elements of  $\cup_{n \in \mathbb{N}} \tau_n$ . So  $O = \cup_{j \in J} (O_{j,1} \cap O_{j,2} \cap \dots \cap O_{j,n_j})$  for some index set  $J$ . Let  $i(j, k) = \min\{n \in \mathbb{N} \mid O_{j,k} \in \tau_n\}$  and  $i(j) = \max\{i(j, k) \mid 1 \leq k \leq n_j\}$ . Then  $O_j = \cap_{1 \leq k \leq n_j} O_{j,k} \in \tau_{i(j)}$  because  $(\tau_n)_{n \in \mathbb{N}}$  is increasing. So  $O = \cup_{j \in J} O_j = \cup_{n \in \mathbb{N}} (\cup_{\{j \mid i(j)=n\}} O_j) = \cup_{n \in \mathbb{N}} O'_n$  where  $O'_n = \cup_{\{j \mid i(j)=n\}} O_j \in \tau_n$ . Therefore, each  $\tau_\infty$ -open set is a countable union of open sets in  $\cup_{n \in \mathbb{N}} \tau_n$ . Since each  $O'_n \in \sigma(\tau_n) = \mathcal{B}_X$ ,

$\tau_\infty \subseteq \mathcal{B}_X$  and  $\sigma(\tau_\infty) \subseteq \mathcal{B}_X$ . On the other hand,  $\tau_0 \subseteq \tau_\infty$  implies  $\mathcal{B}_X = \sigma(\tau_0) \subseteq \sigma(\tau_\infty)$ . Therefore,  $\sigma(\tau_\infty) = \mathcal{B}_X$ .

Finally, continuity of  $K : (X, \tau_\infty) \rightarrow (\mathbb{P}(X), (\tau_n)_{\mathbb{P}(X)})$  follows from that of  $K : (X, \tau_{n+1}) \rightarrow (\mathbb{P}(X), (\tau_n)_{\mathbb{P}(X)})$ , for all  $n \in \mathbb{N}$ .  $\square$

For the next result, we will need to appeal to the famous Portmanteau Theorem, as found for example in [20].

**Theorem 13 (Portmanteau Theorem).** *Let  $P$  and  $(P_n)_{n \in \mathbb{N}}$  be a sequence of probability measures on  $(X, \Sigma)$ , a metric space with its Borel  $\sigma$ -algebra. Then the following five conditions are equivalent:*

1.  $P_n \Rightarrow P$ .
2.  $\liminf_n \int f dP_n = \int f dP$  for all bounded, uniformly continuous real  $f$ .
3.  $\limsup_n P_n(F) \leq P(F)$  for all closed  $F$ .
4.  $\liminf_n P_n(G) \geq P(G)$  for all open  $G$ .
5.  $\lim_n P_n(A) = P(A)$  for all  $P$ -continuity sets  $A$ .

**Lemma 4.** *The least upper bound of the weak topologies  $\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)}$  exists and*

$$\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)} = (\tau_\infty)_{\mathbb{P}(X)}.$$

*Proof.* Again,  $((\tau_n)_{\mathbb{P}(X)})_{n \in \mathbb{N}}$  is an increasing sequence of Polish spaces, and so  $\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)}$  exists. Clearly,  $\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)} \subseteq (\tau_\infty)_{\mathbb{P}(X)}$ .

Suppose  $P_n \Rightarrow P$  in  $\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)}$ . Then  $P_n \Rightarrow P$  in  $(\tau_n)_{\mathbb{P}(X)}$  for all  $n \in \mathbb{N}$ . Let  $O$  be a  $\tau_\infty$ -open set. Then as in the proof of Lemma 3,  $O = \bigcup_{n \in \mathbb{N}} O_n$  where each  $O_n \in \tau_n$ . Let  $G_j = \bigcup_{n=1}^j O_n \in \tau_j \subseteq \mathcal{B}_X$ . Then  $(G_j)_{j \in \mathbb{N}}$  increases to  $O$ . So  $P_n(O) \geq P_n(G_j)$  for all  $n, j \in \mathbb{N}$ . So  $\liminf_n P_n(O) \geq \liminf_n P_n(G_j) \geq P(G_j)$  for all  $j \in \mathbb{N}$  by Theorem 13 in  $(\tau_j)_{\mathbb{P}(X)}$ . So  $\liminf_n P_n(O) \geq \lim_j P(G_j) = P(O)$  by continuity from below. So  $P_n \Rightarrow P$  in  $(\tau_\infty)_{\mathbb{P}(X)}$  by Theorem 13 in  $(\tau_\infty)_{\mathbb{P}(X)}$ . Therefore,  $(\tau_\infty)_{\mathbb{P}(X)} \subseteq \bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)}$  whence equality follows.  $\square$

We are now able to prove the main theorem of this section.

*Proof (Theorem 11).* By Lemmas 2 and 3, there exist Polish topologies  $(\tau_n)_{n \in \mathbb{N}}$  and  $\tau_\infty$  on  $X$ , finer than  $\tau$ , such that  $\sigma(\tau_\infty) = \mathcal{B}_X$  and  $K : (X, \tau_\infty) \rightarrow (\mathbb{P}(X), (\tau_n)_{\mathbb{P}(X)})$  is continuous for all  $n \in \mathbb{N}$ . This is equivalent to continuity of  $K : (X, \tau_\infty) \rightarrow (\mathbb{P}(X), \bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)})$  since convergence in  $\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)}$  is equivalent to convergence in  $(\tau_n)_{\mathbb{P}(X)}$  for all  $n \in \mathbb{N}$  (again this follows from  $\bigvee_{n \in \mathbb{N}} (\tau_n)_{\mathbb{P}(X)}$ -open sets being unions of  $(\tau_n)_{\mathbb{P}(X)}$ -open sets). But then  $K : (X, \tau_\infty) \rightarrow (\mathbb{P}(X), (\tau_\infty)_{\mathbb{P}(X)})$  is continuous by Lemma 4.  $\square$

This argument easily extends to a countable family of stochastic kernels, so that we have the following.

**Corollary 2.** *Let  $(X, \tau)$  be a Polish space and  $(\mathbb{P}(X), \tau_{\mathbb{P}(X)})$  be the space of probability measures on  $X$  equipped with the topology of weak convergence with respect to  $\tau$ . Let  $(K_n)_{n \in \mathbb{N}}$  be a sequence of stochastic kernels on  $X$ . Then there exists a finer Polish topology  $\tau'$  on  $X$  such that  $\sigma(\tau') = \mathcal{B}_X$ ,  $\sigma(\tau'_{\mathbb{P}(X)}) = \mathcal{B}_{\mathbb{P}(X)}$ , and each  $K_n : (X, \tau') \rightarrow (\mathbb{P}(X), \tau'_{\mathbb{P}(X)})$  is continuous.*

**Corollary 3.** *Let  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP. Then there exists a Polish topology  $\tau$  on  $S$  that generates  $\mathcal{B}_S$  and makes  $r_a$  and  $P_a$  continuous for each  $a$  in  $A$ , where  $\mathbb{P}(S)$  is endowed with the topology of weak convergence induced by  $\tau$ .*

Thus, if  $r$  is bounded and measurable we may apply Theorem 7 to obtain a quantitative form of bisimilarity. It is important to keep in mind what is going on here from a practical point of view: if we begin with a modelling scenario in which the rewards are discontinuous with a *given metric* then this amounts to changing that metric to one with respect to which rewards *are* continuous. Therefore the usefulness of this result is contingent on the modelling problem at hand not crucially being dependent on any specific metric.

With that caveat in mind, we now come to the main result of this paper, a general version of Theorem 10.

**Corollary 4.** *Suppose  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  is an MDP and that the image of  $r$  is contained in  $[0, 1]$ . Let  $c \in (0, 1)$  be a discount factor,  $\mathcal{F}_c$  be the family of functional expressions defined in Definition 15, and  $\mathcal{G}$  be a family of functional expressions such that  $\mathcal{F}_c \subseteq \mathcal{G} \subseteq \text{Lip}(\mathcal{F}_c)$ . Then the pseudometric induced by  $\mathcal{G}$  is the unique fixed-point  $\rho_c^*$  satisfying the equation*

$$\rho_c^*(x, y) = \max_{a \in A} ((1 - c)|r_a(x) - r_a(y)| + c \cdot \mathfrak{K}(\rho_c^*)(P_a(x), P_a(y))) \text{ for all } x, y \in S$$

*and whose kernel is bisimilarity.*

## 4 Estimating Bisimulation

In this section, we discuss how focusing on families of functional expressions may make estimating bisimilarity more amenable in practice. Assume we are given an MDP  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  where the image of  $r$  is contained in  $[0, 1]$ . Computing a bisimilarity metric for a finite  $\mathcal{M}$  has encompassed estimating the integral probability metric  $\mathcal{F}_c(P, Q)$ , yielding an algorithm with exponential complexity [9], computing the Kantorovich metric,  $\mathfrak{K}(\cdot)(P, Q)$ , yielding an algorithm with polynomial complexity [3], and solving a linear program [30].

The major issue is that although computing the linear programming formulations of bisimilarity in the ideal case can be done in polynomial time, to do so in practice is highly inefficient; to understand why, one may remark that the linear programs for a given MDP are more complex than solving for the discounted value function for that MDP; although the latter is also known to be solvable in polynomial time by linear programming [31], in practice Monte Carlo techniques have been found to be much more successful. In fact, in [4], we focused on estimating the Kantorovich metric by replacing each  $P$  and  $Q$  by empirical measures; this idea is studied in better depth in [32]. We will not focus on that approach here.

Instead we focus on a heuristic approach implicitly used in [17] and Monte Carlo techniques used in [33]. In the former, the problem at hand is, given a distribution over MDPs with a common state space, to try to find a policy that optimizes the expected total geometrically-discounted sum of rewards achieved at each state, where the average is taken over a number of sample runs performed on a number of MDPs drawn according to the given MDP distribution. The authors attack this problem by generating a family of functional expressions according to some distribution, and using these to estimate optimal planning strategies - the so-called *formula-based exploration / exploitation strategies*. In [33], the authors solve the problem of trying to compute the infimum over a given set by instead sampling and then estimating the essential infimum. Since in our case we are interested in suprema, let us recall the definition of essential supremum.

**Definition 16.** Let  $(X, \mathcal{B}_X, \mu)$  be a measure space. The essential supremum of a bounded measurable function  $f : (X, \mathcal{B}_X) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$  is given by the following.

$$\text{ess sup } f = \inf\{\alpha \in \mathbb{R} : \mu(\{x \in X : f(x) > \alpha\}) = 0\}.$$

In other words,  $\text{ess sup } f$  is the least real number that is an upper bound on  $f$  except for a set of  $\mu$ -measure zero. It follows that in general,  $\text{ess sup } f \leq \sup f$ . Suppose further that  $\mathcal{B}_X$  is a Borel  $\sigma$ -algebra,  $f$  is continuous, and  $\mu$  is a *strictly positive measure*, i.e. every non-empty open subset of  $X$  has strictly positive  $\mu$ -measure. Then since  $\{x \in X : f(x) > \alpha\} = f^{-1}(\alpha, \infty)$  is open, it follows that it has  $\mu$ -measure zero if and only if it is the empty set; in this case, the essential supremum and the supremum agree. We will use this in conjunction with Lemma 2 from [33], restated here in terms of the essential supremum in place of the essential infimum.

**Lemma 5.** Let  $(\Omega, \Sigma, \mathbb{P})$  and  $(X, \mathcal{B}_X, \mu)$  be probability spaces and assume that we can sample random variables  $X_1, X_2, \dots, X_n$  mapping  $\Omega$  to  $X$ , independently and identically distributed according to  $\mu$ . Then if  $f : X \rightarrow \mathbb{R}$  is bounded and measurable we have

$$\max_{1 \leq i \leq n} f(X_i) \rightarrow \text{ess sup } f \text{ in } \mu\text{-probability as } n \rightarrow \infty. \quad (10)$$

This allows for another Monte Carlo technique for (under)approximating the Kantorovich metric for bisimilarity in an MDP.

**Proposition 2.** Let  $\mathcal{M} = (S, \mathcal{B}_S, A, \{P_a : a \in A\}, r)$  be an MDP where the image of  $r$  is contained in  $[0, 1]$ . Let  $c \in (0, 1)$  be a discount factor and  $\mathcal{F}_c$  be the family of functional expressions defined in Definition 15 and interpreted over  $\mathcal{M}$ . Let  $\overline{\mathcal{F}_c}$  be the closure of  $\mathcal{F}_c$  with respect to uniform convergence. Let  $\mu \in \mathbb{P}(\overline{\mathcal{F}_c})$  be strictly positive. Suppose  $f_1, f_2, \dots, f_n$  are independent, identically distributed samples drawn according to  $\mu$ . Then

$$\max_{1 \leq i \leq n} |P(f_i) - Q(f_i)| \rightarrow \mathcal{F}_c(P, Q) \text{ in } \mu\text{-probability as } n \rightarrow \infty. \quad (11)$$

*Proof.* Since  $S$  is Hausdorff,  $\mathbb{C}^b(S)$  with the uniform norm is a Banach space. Therefore,  $\overline{\mathcal{F}_c}$ , as a closed subset of  $\mathbb{C}^b(S)$ , is itself a measurable subspace when equipped with the Borel sets given by the uniform norm. For a given  $P, Q \in \mathbb{P}(S)$  let  $g : \overline{\mathcal{F}_c} \rightarrow \mathbb{R}$  be defined by  $g(f) = |P(f) - Q(f)|$ . Then  $g$  is continuous and bounded by 1. The result now follows from Lemma 5 and the preceding remarks, and Corollary 4.  $\square$

Remark in particular that

$$\max_{1 \leq i \leq n} |f_i(x) - f_i(y)| \rightarrow \rho_c^*(x, y) \text{ in } \mu\text{-probability as } n \rightarrow \infty. \quad (12)$$

To turn this into a proper algorithm is beyond the scope of this work - one needs to fix a particular measure and provide sample complexity results, among other things. However, we remark that being able to sample from a much smaller class than the class of all Lipschitz functions should improve performance regardless of how other parameters are set.

## 5 Conclusions

We have shown, with slight modification, that the family of functional expressions developed in [9, 2] to capture quantitative bisimilarity for LMPs does the same for MDPs with continuous state spaces and bounded measurable reward functions. We have used the same techniques as in these previous works - in particular, a density result in Proposition 2 of [2] - reworded in the terminology of generating classes for integral probability metrics. The hope is that by focusing on these generating classes of functions, we may find better practical algorithms for assessing equivalence between states in a Markov process - either by under or over-approximating a particular class, or by sampling from it in some manner.

Moreover, we have used a trick from descriptive set theory to remove a previous continuity constraint on the rewards and Markov kernels in Theorem 7, thereby widening its applicability.

### 5.1 Related Work

The notion of bisimilarity metrics, both in terms of logical expressions and in terms of how to compute them using linear programming formulations, really derives from the work of [9] and [2] for LMPs. In [9], the emphasis was on developing a robust theoretical notion of quantitative bisimilarity and establishing a decision procedure for it, albeit with exponential complexity. In [2], the emphasis was again on establishing a robust notion of quantitative bisimilarity while at the same time yielding a theoretical polynomial complexity bound by means of the Kantorovich metric. Complexity results in general are discussed in [30]. However, in none of these are more than a few toy examples worked through, and the idea of Monte Carlo techniques for more efficient practical implementations is not broached.

The idea of examining the relationship between probability measures by studying generating classes of functions was explored in [5, 34] for integral probability metrics and stochastic orders. Müller takes the point of view of looking at maximal generators for such orders, and demonstrates that in general, minimal orders may not exist.

To the best of our knowledge, the only practical work to exploit optimality based on functional expression occurs in [17]. Here, the goal is to determine an optimal planning strategy on average, when one is acting on an unknown MDP but given a distribution over its reward and transition parameters. The advantage of the functional expression approach here is that it is independent of the particulars of a given model.

### 5.2 Future Work

The point of view of this work is that one should focus on families of functional expressions for quantitative bisimilarity as we suspect this may be more advantageous in practice. Thus, an immediate concern is to turn Proposition 2 into a full-fledged Monte Carlo algorithm. Among the necessities are choosing the right class of functional expressions from which to sample, as small as possible a subset of  $\mathcal{F}_c$ , constructing a strictly positive probability measure with which to sample the class of functionals, and most importantly, a sample complexity bound to inform us of how many samples should be required for a given level of confidence.

From the theoretical side, we are interested in finding minimal classes that generate the same bisimilarity metric, and equivalences obtained from using other classes. In both cases, it might be fruitful to consider only non-empty closed subsets of  $\mathbb{C}^b(S)$  with the uniform norm. We can order

this space, and add in the empty set, to get a complete lattice; moreover, we can equip it with the Hausdorff metric, and the resulting Borel  $\sigma$ -algebra, known as the *Effros Borel space*, will be a standard Borel space provided  $(S, \mathcal{B}_S)$  is as well ([27], pg. 97). Doing so may allow us to relate the differences between the equivalences induced by two families of functional expressions in terms of their quantitative difference in Effros Borel space. In particular, we are interested in coarser more easily computable equivalences, and how to relate these to the theoretical guarantees given by bisimilarity.

In statistical parlance, the interpreted class of functional expressions is just a family of random variables; and testing whether or not two states are bisimilar amounts to testing how their Markov kernels differ on this test set of random variables. Conceptually, this fits in with Prakash's view that Markov processes should be viewed as transformers of random variables [35]. As (real-valued) stochastic kernels subsume both random variables and subprobability measures, we may complete this conceptual picture by viewing a Markov process - itself a family of kernels - as a transformer of families of kernels. It remains to be seen if this point of view in general can lead to better algorithms in practice.

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