Steerable Derivatives

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Steerable filters in the current context represent the realization of an anisotropic filter at any angle θ by a linear combination of n distinct angular instances of the filter. In this note the formulation of one-dimensional derivatives as steerable filters is summarized. For further literature on steerable filters see [1, 2, 3, 4, 5].

The simplest steerable derivative filter is the first derivative, also known as the directional derivative, given in *operator notation*[1] for ease of exposition by,

$$\mathcal{D}_\theta^1 = \mathcal{D}_x \cos(\theta) + \mathcal{D}_y \sin(\theta). \tag{2}$$

More generally, the steerable higher-order derivative of order p is given by,

$$\begin{aligned}
\mathcal{D}_\theta^p &= (\mathcal{D}_x \cos(\theta) + \mathcal{D}_y sin(\theta))^p & (3)\\
&= \sum_{k=0}^{p} \binom{p}{k} \Big(\mathcal{D}_x \cos(\theta)\Big)^k \Big(\mathcal{D}_y sin(\theta)\Big)^{p-k} \qquad \{\text{binomial theorem}\} & (4)\\
&= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \Big(\mathcal{D}_x \cos(\theta)\Big)^k \Big(\mathcal{D}_y sin(\theta)\Big)^{p-k}. & (5)
\end{aligned}$$

The direction of the maximum response of $\mathcal{D}_{\theta_{\max}}^p$ may be found by differentiating (5), setting the result to zero and solving for θ. In the case of $\mathcal{D}_{\theta_{\max}}^1$ this process yields the gradient direction,

$$\theta_{\max} = \operatorname{atan}\left(\frac{\mathcal{D}_y}{\mathcal{D}_x}\right) \tag{6}$$

[1]In operator notation the convolution operation is written as,

$$\mathbf{I}' = \mathcal{T}\mathbf{I} \tag{1}$$

where $\mathcal{T}$ represents the convolution operator (denoted in calligraphic lettering) which maps the image $\mathbf{I}$ into the image $\mathbf{I}'$.

and for $\mathcal{D}^2_{\theta_{\max}}$ yields the direction of maximum curvature,

$$\theta_{\max} = \frac{1}{2}\text{atan}\left(\frac{2\mathcal{D}_{xy}}{\mathcal{D}_{yy} - \mathcal{D}_{xx}}\right). \tag{7}$$

References

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