The Analysis of Multiple Motions using the Principle of Superposition

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In this note we briefly summarize Shizawa and Mase's [2] computational framework for the analysis of multiple motions. This framework specifically addresses the case of additive transparency; the superposition of multiple translating images. The derivation presented next is adapted from [1] which omits the tensor notation used in [2].

Let's begin with the case of the superposition of two translating signals, $f_1(\mathbf{x}, t)$ and $f_2(\mathbf{x}, t)$, with image velocity, $\mathbf{v}_1 = (u_1, v_1)$ and $\mathbf{v}_2 = (u_2, v_2)$, which are assumed to be locally constant. Their combination in the form of additive transparency is defined as follows,

$$f(\mathbf{x}, t) = f_1(\mathbf{x}, t) + f_2(\mathbf{x}, t) \tag{1}$$

Each of the translating images adheres to the optical flow constraint,

$$(\mathbf{v}_i, 1) \cdot \nabla f_i(\mathbf{x}, t) = 0, \quad i = 1, 2 \tag{2}$$

where $((\mathbf{v}, 1) \cdot \nabla) = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial}{\partial t})$ denotes a differential operator.

Beginning with (1) apply the differential operators to both sides, as follows,

$$((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla) f(\mathbf{x}, t) = ((\mathbf{v}_1, t) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla)[f_1(\mathbf{x}, 1) + f_2(\mathbf{x}, t)] \tag{3}$$

Next, for clarity of presentation, let L and R denote the left and right hand sides of (3), respectively,

$$L = ((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla) f(\mathbf{x}, t) \tag{4}$$

$$R = ((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla)[f_1(\mathbf{x}, t) + f_2(\mathbf{x}, t)] \tag{5}$$

$$= ((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla) f_1(\mathbf{x}, t) + ((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla) f_2(\mathbf{x}, t) \tag{6}$$

Since $\mathbf{v}_i$ are constants, the differential operators commute,

$$R = ((\mathbf{v}_2, 1) \cdot \nabla)((\mathbf{v}_1, 1) \cdot \nabla) f_1(\mathbf{x}, t) + ((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla) f_2(\mathbf{x}, t) \tag{7}$$

Substituting constraints (2) into (7), yields,

$$R = ((\mathbf{v}_2, 1) \cdot \nabla)0 + ((\mathbf{v}_1, 1) \cdot \nabla)0 \quad (8)$$
$$= 0 \quad (9)$$

Putting L and R together forms the following constraint,

$$((\mathbf{v}_1, 1) \cdot \nabla)((\mathbf{v}_2, 1) \cdot \nabla) f(\mathbf{x}, t) = 0 \quad (10)$$

or

$$u_1 u_2 f_{xx} + v_1 v_2 f_{yy} + f_{tt} + (u_1 v_2 + u_2 v_1) f_{xy} + (u_1 + u_2) f_{xt} + (v_1 + v_2) f_{yt} = 0 \quad (11)$$

The authors demonstrate that using differential measures of $f(\mathbf{x}, t)$ at four or five spatial positions yields closed-form solutions for the velocities $\mathbf{v}_i$. The estimation of n translating images is an extension of (11) that includes higher order differential operators, formally,

$$((\mathbf{v_1}, 1) \cdot \nabla)((\mathbf{v_2}, 1) \cdot \nabla) \cdots ((\mathbf{v_n}, 1) \cdot \nabla) f(\mathbf{x}, t) = 0 \quad (12)$$

Note due to the commutativity of the differential operators their order in (12) does not matter.

Finally, inspection of constraint (11) in the Fourier domain reveals that the solution is equivalent to fitting planes to the power spectrum of $f(\mathbf{x}, t)$ [1].

References

[1] K. Langley, D.J. Fleet, and T.J. Atherton. Multiple motions from instantaneous frequency. In *IEEE Conference on Computer Vision and Pattern Recognition*, pages 846–849, 1992.

[2] M. Shizawa and K. Mase. Principle of superposition: A common computational framework foranalysis of multiple motion. In *Motion Workshop*, pages 164–172, 1991.