

Parameterizing a Non-Axis Aligned Ellipse

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Version 1.0

February 1, 2005

The following note presents the parameterization of a non-axis aligned ellipse, represented as follows,

$$\alpha x^2 + \beta xy + \gamma y^2 = 1 \quad (1)$$

or equivalently,

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 1 \quad (2)$$

where

$$A = \begin{bmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{bmatrix} \quad (3)$$

$\alpha, \gamma > 0$ and $\beta \neq 0$. In the case where $\beta = 0$, the ellipse is axis aligned.

Since A is *positive definite*¹ [1]: $A = Q\Lambda Q^\top$, where the columns of Q contain the orthonormal eigenvectors of A and Λ the corresponding eigenvalues along the diagonal ($\lambda_{1,2} > 0$) [1], (2) can be rewritten as follows:

$$\mathbf{x}^\top Q \Lambda Q^\top \mathbf{x} = 1 \quad (4)$$

$$(Q^\top \mathbf{x})^\top \Lambda (Q^\top \mathbf{x}) = 1 \quad (5)$$

Let $Q^\top \mathbf{x} = \mathbf{y}$, substituting into (5), yields,

$$\mathbf{y}^\top \Lambda \mathbf{y} = 1 \quad (6)$$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1. \quad (7)$$

The change from $\mathbf{x}$ to $\mathbf{y}$ amounts to rotating the non-axis aligned ellipse (2) by $Q^\top$, such that the major and minor axes are aligned with the coordinate axes (see Fig. 1).

Given the axis aligned description of the ellipse, the ellipse can be parameterized in the standard way, as follows,

$$[y_1, y_2]^\top = \left[\frac{1}{\sqrt{\lambda_1}} \cos(\theta), \frac{1}{\sqrt{\lambda_2}} \sin(\theta) \right] \quad (8)$$

where $0 \leq \theta < 2\pi$.

¹Definition: Matrix B is positive definite if $\mathbf{x}^\top B \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

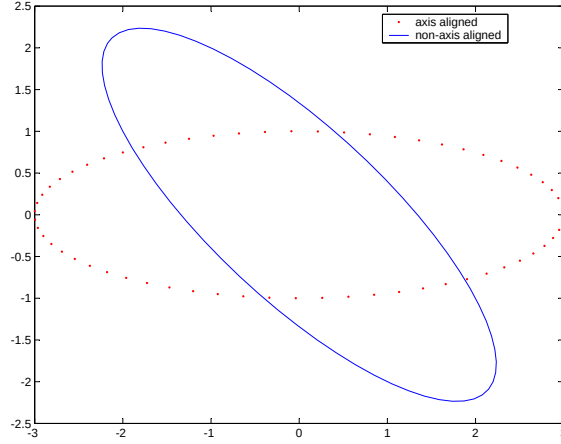


Figure 1: Depicted are plots of the initial non-axis aligned ellipse and its derotated axis aligned pairing.

Finally, the parameterization of the original non-axis aligned ellipse is realized by applying the rotation Q to (8),

$$\mathbf{x} = Q \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (9)$$

Example: Find the parameterization of: $\frac{5}{9}x^2 + \frac{8}{9}xy + \frac{5}{9}y^2 = 1$.

Answer: Beginning with

$$\frac{5}{9}x^2 + \frac{8}{9}xy + \frac{5}{9}y^2 = 1 \quad (10)$$

rewrite as,

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \quad (11)$$

factor $\begin{bmatrix} \frac{5}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{5}{9} \end{bmatrix}$ into $Q\Lambda Q^\top$, where,

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad (12)$$

$$\Lambda = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 1 \end{bmatrix} \quad (13)$$

Finally, the parameterization of the non-axis ellipse is given as,

$$\begin{bmatrix} x \\ y \end{bmatrix} = Q \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} \cos(\theta) \\ \frac{1}{\sqrt{\lambda_2}} \sin(\theta) \end{bmatrix} \quad (14)$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1/9}} \cos(\theta) \\ (1) \sin(\theta) \end{bmatrix} \quad (15)$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2/3}} \cos(\theta) + \frac{1}{\sqrt{2}} \sin(\theta) \\ \frac{1}{\sqrt{2/3}} \cos(\theta) + \frac{1}{\sqrt{2}} \sin(\theta) \end{bmatrix} \quad (16)$$

References

- [1] Gilbert Strang. *Linear Algebra and its Applications*. Harcourt Brace and Company, third edition, 1988.