Affine motion: Kinematic parameter definitions

Konstantinos G. Derpanis
kosta@cs.yorku.ca

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For a small image region, an affine transformation can provide an accurate approximation of the image motion of a smooth surface. This model can be written as

$$\vec{u} = \mathbf{A}\vec{x} + \vec{t}, \tag{1}$$

where

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix}, \tag{2}$$

$\vec{u} = (u, v)^\top$ represents the 2D displacement, $\vec{x} = (x, y)^\top$ represents the point and $\vec{t} = (a_0, a_3)^\top$ represents the x and y translational components respectively.

In understanding the ramifications of the the transformation embodied in $\mathbf{A}$, it is advantageous to recast it in terms of kinematic quantities that capture (infinitesimal) rotation (curl), expansion/contraction (divergence), and shear (deformation) ((Aris, 1989; Koenderink & van Doorn, 1976; Longuet-Higgins & Pradzny, 1980; Waxman & Ullman, 1985)). Toward that end $\mathbf{A}$ can be rewritten as a sum of a symmetric and antisymmetric matrix as follows,

$$\mathbf{A} = \frac{1}{2}[(\mathbf{A} + \mathbf{A}^\top) + (\mathbf{A} - \mathbf{A}^\top)]. \tag{3}$$

The antisymmetic part can be expressed as,

$$\mathbf{A} - \mathbf{A}^\top = \begin{pmatrix} 0 & a_2 - a_4 \\ -a_2 + a_4 & 0 \end{pmatrix} = (-a_2 + a_4) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

The symmetric part can be further decomposed into two components, the sum of a scalar multiple of the identity matrix and a symmetric matrix

$$\mathbf{A} + \mathbf{A}^\top = \begin{pmatrix} 2a_1 & a_2 + a_4 \\ a_2 + a_4 & 2a_5 \end{pmatrix} = \begin{pmatrix} a_1 + a_5 & 0 \\ 0 & a_1 + a_5 \end{pmatrix} + \begin{pmatrix} a_1 - a_5 & a_2 + a_4 \\ a_2 + a_4 & -a_1 + a_5 \end{pmatrix}. \tag{5}$$

With the above decompositions, $\mathbf{A}$ can be expressed as,

$$\mathbf{A} = \frac{1}{2} curl \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} div \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} def \mathbf{S}, \tag{6}$$

where

$$\begin{array}{rcl} div & = & a_1 + a_5 \\ curl & = & -a_2 + a_4 \\ def & = & \sqrt{(a_1 - a_5)^2 + (a_2 + a_4)^2} \end{array}, \tag{7}$$

and **S** is a traceless matrix defining the direction of the area preserving deformation. It is interesting to note that the divergence, curl and magnitude of the deformation are invariant to rotations of the image coordinate frame.

References

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