

Gabor filters

Konstantinos G. Derpanis

York University

kosta@cs.yorku.ca

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In this note the Gabor filter is reviewed. The Gabor filter was originally introduced by Dennis Gabor ([Gabor, 1946](#)). The one-dimensional Gabor filter is defined as the multiplication of a cosine/sine (even/odd) wave with a Gaussian windows (see [Fig. 1](#)), as follows,

$$g_e(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cos(2\pi\omega_0 x) \quad (1)$$

$$g_o(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \sin(2\pi\omega_0 x) \quad (2)$$

where ω_0 defines the centre frequency (i.e., the frequency in which the filter yields the greatest response) and σ the spread of the Gaussian window.

The power spectrum of the Gabor filter is given by the sum of two Gaussians centred at $\pm\omega_0$:

$$\|G(\omega)\| = e^{-2\pi^2\sigma^2(\omega-\omega_0)^2} + e^{-2\pi^2\sigma^2(\omega+\omega_0)^2} \quad (3)$$

This can be reasoned as follows. The power spectrum of a sine wave are two impulses located at $\pm\omega_0$ and the power spectrum of Gaussian is a (non-normalized) Gaussian. Multiplication in the temporal (spatial) domain is equivalent to convolution in the frequency domain ([Oppenheim, Willsky & S.H., 1997](#)).

The *uncertainty principle* states that the product of the spread (i.e., uncertainty) of a signal in the time and frequency domains must exceed or equal a fixed constant,

$$\Delta t \Delta f = c, \quad (4)$$

where c ¹ is a constant, Δt and Δf represent the measure of the spread of the signal in the time and frequency domains, respectively (see [Fig. 2](#)). The implication of this

¹The exact value of the constant c depends on the form of the Fourier transform used.

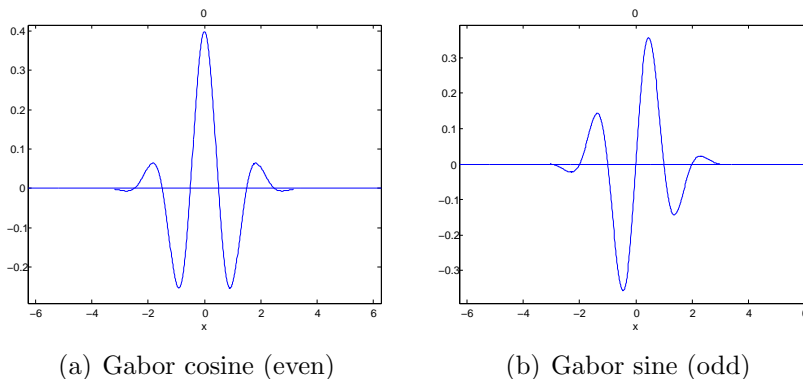


Figure 1: One-dimensional Gabor filters.

principle is that the accuracy with which one can measure a signal in one domain limits the attainable accuracy of the measurement in the other domain. Gabor (Gabor, 1946) demonstrated that the **complex** Gabor filter given by,

$$\begin{aligned}
 g(x) &= g_e(x) + i g_o(x) \\
 &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \left(\cos(2\pi\omega_0 x) + i \sin(2\pi\omega_0 x) \right) \\
 &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} e^{i(2\pi\omega_0 x)}
 \end{aligned} \tag{5}$$

attains the optimal (lower bound) compromise between the localization in the time and frequency domains; notice that the Gaussian function is an instance of a Gabor filter with centre frequency $\omega_0 = 0$. Note that the constituent real-valued Gabor elementary functions (i.e., the even and odd part taken separately) do **not** as widely believed minimize the joint uncertainty (Stork & Wilson, 1990). Furthermore, the selection of a different localization measure may result in a different class of “optimal” function altogether (Lerner, 1961; Stork & Wilson, 1990), casting doubt on the primacy of the Gabor function often cited in the literature.

Daugman (Daugman, 1980; Daugman, 1985) extended the Gabor filter to two-dimensions (see Fig. 3), as follows,

$$g_e(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)} \cos(2\pi\omega_{x_0}x + 2\pi\omega_{y_0}y) \tag{6}$$

$$g_o(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)} \sin(2\pi\omega_{x_0}x + 2\pi\omega_{y_0}y) \tag{7}$$

where $(\omega_{x_0}, \omega_{y_0})$ defines the centre frequency and (σ_x, σ_y) the (potentially asymmetric) spread of the Gaussian window.

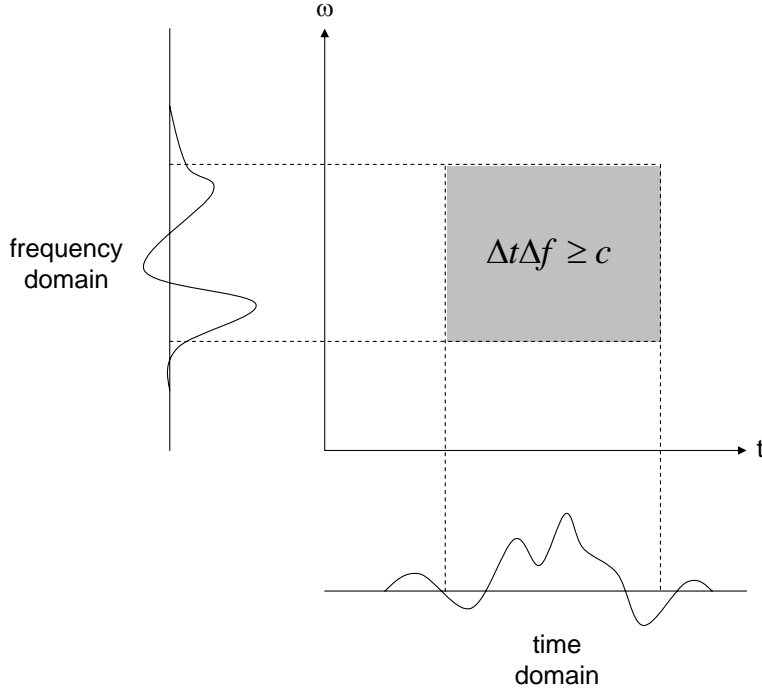


Figure 2: Joint localization of a signal in time and frequency domains.

For the purpose of extracting optical flow, Heeger (Heeger, 1987) utilized the three-dimensional (space-time) Gabor filter,

$$g_e(x, y, t) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_t} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{t^2}{\sigma_t^2}\right)} \cos(2\pi\omega_{x_0}x + 2\pi\omega_{y_0}y + 2\pi\omega_{t_0}t) \quad (8)$$

$$g_o(x, y, t) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_t} e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} + \frac{t^2}{\sigma_t^2}\right)} \sin(2\pi\omega_{x_0}x + 2\pi\omega_{y_0}y + 2\pi\omega_{t_0}t) \quad (9)$$

where $(\omega_{x_0}, \omega_{y_0}, \omega_{t_0})$ defines the centre frequency and $(\sigma_x, \sigma_y, \sigma_t)$ defines the (potentially asymmetric) spread of the Gaussian window.

Heeger (Heeger, 1987) demonstrated that the three-dimensional (similarly for the two-dimensional case) Gabor filter can be built from one-dimensional separable components. Considering the two-dimensional Gabor filter, let k be the size of the two-dimensional convolution kernel and n be the size of an image (in pixels). The complexity of the non-separable convolution of the Gabor filter is reduced from $O(k^2n^2)$ to $O(kn^2)$.

An application of Gabor filters is in local time-frequency analysis of signals, specifically, a fixed windowed *Fourier transform*, referred to as the *Gabor transform*. A difficulty with the Gabor transform is that it is linearly independent but highly non-orthogonal and as such cannot be easily inverted. As a result of non-orthogonality,

the functions, $r_i(n)$, used for reconstructing the discrete signal, $f(n)$, are highly distinct from the Gabor functions used to recover the coefficients of the representation (i.e., the analysis step), where the coefficients indicate how much of its corresponding reconstruction filter, $r_i(n)$, is to be added, formally,

$$f(n) = \sum_i c_i r_i(n) \quad \text{reconstruction step} \quad (10)$$

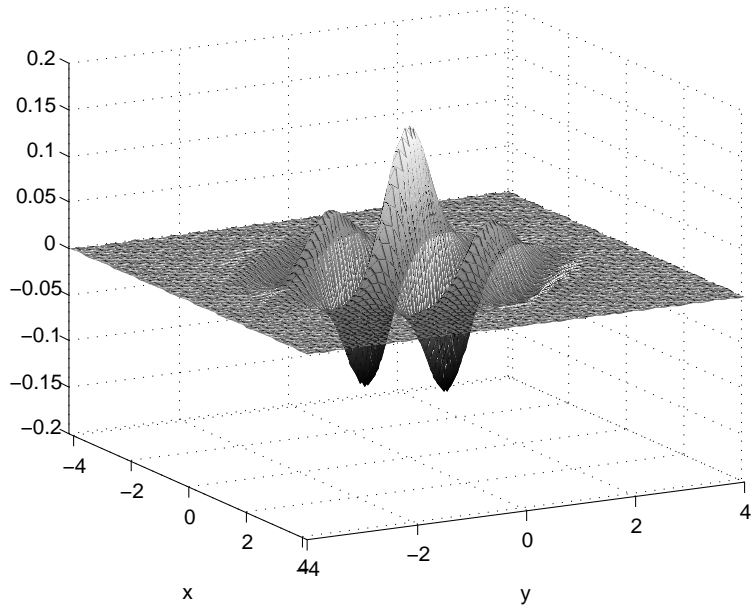
where,

$$c_i = \sum_n f(n) g_i(n) \quad \text{analysis step.} \quad (11)$$

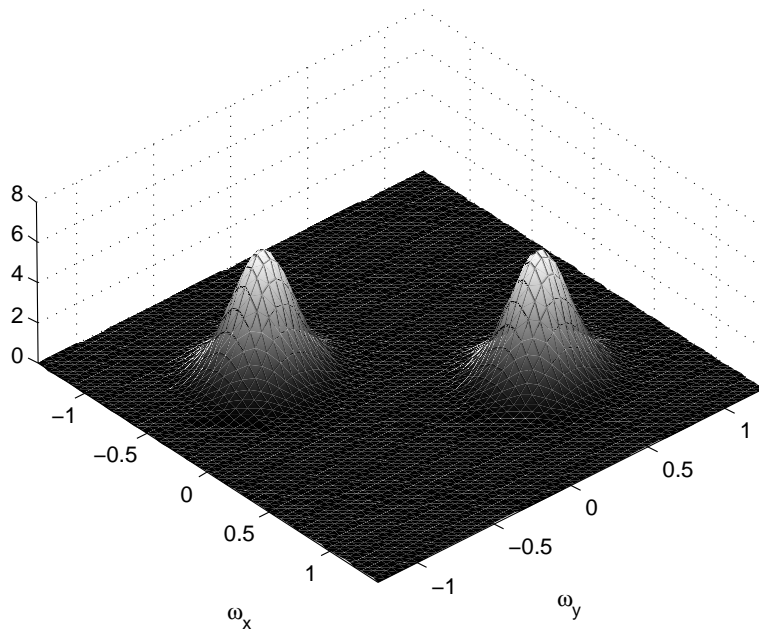
If the Gabor transform were indeed orthogonal (such as the *Fourier transform*), $g_i(n) = r_i(n)$.

References

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(a) Gabor cosine (even)



(b) Gabor power spectrum

Figure 3: Two-dimensional Gabor filters.