Fourier Transform of the Gaussian

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In this note we consider the Fourier transform[1] of the Gaussian.
The Gaussian function, $g(x)$, is defined as,

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}, \tag{3}$$

where $\int_{-\infty}^{\infty} g(x)dx = 1$ (i.e., normalized). The Fourier transform of the Gaussian function is given by:

$$G(\omega) = e^{-\frac{\omega^2\sigma^2}{2}}. \tag{4}$$

Proof:
We begin with differentiating the Gaussian function:

$$\frac{dg(x)}{dx} = -\frac{x}{\sigma^2} g(x) \tag{5}$$

Next, applying the Fourier transform to both sides of (5) yields,

$$i\omega G(\omega) = \frac{1}{i\sigma^2} \frac{dG(\omega)}{d\omega} \tag{6}$$

$$\frac{\frac{dG(\omega)}{d\omega}}{G(\omega)} = -\omega\sigma^2. \tag{7}$$

Integrating both sides of (7) yields,

$$\int_0^\omega \frac{\frac{dG(\omega')}{d\omega'}}{G(\omega')} d\omega' = -\int_0^\omega \omega'\sigma^2 d\omega' \tag{8}$$

$$\ln G(\omega) - \ln G(0) = \frac{\sigma^2\omega^2}{2}. \tag{9}$$

Since the Gaussian is normalized, the DC component $G(0) = 0$, thus (9) can be rewritten as,

$$\ln G(\omega) = -\frac{\sigma^2\omega^2}{2} \tag{10}$$

Finally, applying the exponent to each side yields,

$$e^{\ln G(\omega)} = e^{-\frac{\sigma^2\omega^2}{2}} \tag{11}$$

$$G(\omega) = e^{-\frac{\sigma^2\omega^2}{2}} \tag{12}$$

as desired.

[1] The Fourier transform pair is given by:

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx \tag{1}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega x} d\omega, \tag{2}$$

where i denotes the complex unit.