General Proof of the Smoothness of Simoncelli's Donut Operator

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In this note, a general proof of the smoothness of Simoncelli's [1] "donut operator" is presented. Without loss of generality, the two-dimensional image case will be considered.

Claim: The level curves of the sum of the power responses of $N+1$ Nth derivative of Gaussians forms a smooth "donut" in the power spectrum. Or equivalently, the resulting summed energy measure is dependent on radial frequency and independent of angular frequency.

Formally, the summed energy response is given by,

$$\epsilon(\omega_x, \omega_y) = \sum_{n=0}^{N} \left| [\hat{\mathbf{d}}_n \cdot (\omega_x, \omega_y)^\top]^N G(\omega_x, \omega_y) \right|^2, \tag{1}$$

where $G(\omega_x, \omega_y)$ represents the spectrum of the Gaussian, N denotes the derivative order and $\hat{\mathbf{d}}_n$ represents the unit direction vector defined as,

$$\hat{\mathbf{d}}_n = (\cos\theta_n, \sin\theta_n)^\top, \quad \text{where } \theta_n = \frac{2\pi n}{N+1}, 0 \le n \le N. \tag{2}$$

Proof:

$$\begin{aligned}
\epsilon(\omega_x, \omega_y) &= \sum_{n=0}^{N} \left| [\hat{\mathbf{d}}_n \cdot (\omega_x, \omega_y)^\top]^N G(\omega_x, \omega_y) \right|^2 && (3)\\
&= \sum_{n=0}^{N} [\hat{\mathbf{d}}_n \cdot (\omega_x, \omega_y)^\top]^{2N} G(\omega_x, \omega_y)^2 && (4)\\
&= \sum_{n=0}^{N} [\hat{\mathbf{d}}_n \cdot (\|\omega\| \hat{\omega})]^{2N} G(\omega_x, \omega_y)^2 && (5)\\
&= \sum_{n=0}^{N} [\hat{\mathbf{d}}_n \cdot \hat{\omega}]^{2N} [\|\omega\|^N G(\omega_x, \omega_y)]^2. && (6)
\end{aligned}$$

Since the last (square) bracketed term is only a function of the radial component, we can omit it from further consideration.

$$
\begin{aligned}
\epsilon'(\omega_x,\omega_y) &= \sum_{n=0}^{N}[\hat{\mathbf{d}}_n\cdot\hat{\omega}]^{2N} && (7)\\
&= \sum_{n=0}^{N}\left[\cos\left(\phi-\frac{2\pi n}{N+1}\right)\right]^{2N} && (8)\\
&= \frac{1}{2^{2N}}\sum_{n=0}^{N}\left[\exp\left(j(\phi-\frac{2\pi n}{N+1})\right)+\exp\left(-j(\phi-\frac{2\pi n}{N+1})\right)\right]^{2N} && (9)\\
&= \frac{1}{2^{2N}}\sum_{n=0}^{N}\sum_{k=0}^{2N}\binom{2N}{k}\exp\left(jk(\phi-\frac{2\pi n}{N+1})\right)\exp\left(-j(2N-k)(\phi-\frac{2\pi n}{N+1})\right) && (10)\\
&= \frac{1}{2^{2N}}\sum_{k=0}^{2N}\binom{2N}{k}\sum_{n=0}^{N}\exp\left(2j(k-N)(\phi-\frac{2\pi n}{N+1})\right) && (11)\\
&= \frac{1}{2^{2N}}\left[\binom{2N}{k}\sum_{n=0}^{N}\exp(j0)+\sum_{k=0,k\neq N}^{2N}\binom{2N}{k}\sum_{n=0}^{N}\exp\left(2j(k-N)(\phi-\frac{2\pi n}{N+1})\right)\right] && (12)\\
&= \frac{1}{2^{2N}}\left[\binom{2N}{N}(N+1)+\sum_{k=0,k\neq N}^{2N}\binom{2N}{k}\sum_{n=0}^{N}\exp\left(2j(k-N)(\phi-\frac{2\pi n}{N+1})\right)\right], && (13)
\end{aligned}
$$

where ϕ represents the angular frequency of $(\omega_x,\omega_y)^\top$. Since $(k-N)$ is an integer, the second term corresponds to the discrete Fourier transform (DFT) of a constant. As such, the second term is equal to zero. Thus, the remaining portion is dependent only on the radial frequency, as sought.

References

[1] E.P. Simoncelli. Distributed analysis and representation of visual motion. In *MIT Ph.D.*, 1993.