Representing Frequency Content in Complex Notation

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In this note the complex representation of frequency content is reviewed[1].
We will use the following identities,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{2}$$

Through some algebraic manipulation (addition and subtraction), (1) and (2) can be rewritten as,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{3}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \tag{4}$$

Any continuous periodic can be represented as a liinear combination of sines and cosines, formally,

$$f(t) = \sum_{k=1}^{N} A_k \cos 2\pi\omega_k t + B_k \sin 2\pi\omega_k t. \tag{5}$$

We now rewrite (5) in complex form,

$$\begin{aligned}
f(t) &= \sum_{k=1}^{N} A_k \left(\frac{e^{i2\pi\omega_k t} + e^{-i2\pi\omega_k t}}{2} \right) + B_k \left(\frac{e^{i2\pi\omega_k t} - e^{-i2\pi\omega_k t}}{2} \right) && (6)\\
&= \sum_{k=1}^{N} \frac{A_k}{2} e^{i2\pi\omega_k t} + \frac{A_k}{2} e^{-i2\pi\omega_k t} + \frac{B_k}{2} e^{i2\pi\omega_k t} - \frac{B_k}{2} e^{-i2\pi\omega_k t} && (7)\\
&= \sum_{k=1}^{N} \frac{A_k + B_k}{2} e^{i2\pi\omega_k t} + \frac{A_k - B_k}{2} e^{-i2\pi\omega_k t} && (8)\\
&= \sum_{k=1}^{N} C_k e^{i2\pi\omega_k t} + C_{-k} e^{-i2\pi\omega_k t} && (9)\\
& && (10)
\end{aligned}$$

[1] This presentation is adapted from [1]

where

$$C_k = \frac{A_k - iB_k}{2} \tag{11}$$

$$C_{-k} = \frac{A_k + iB_k}{2} \tag{12}$$

$$\tag{13}$$

More compactly,

$$f(t) = \sum_{k=-N}^{N} C_k e^{i2\pi\omega_k t} \tag{14}$$

where

$$C_k = \frac{A_k + iB_k}{2}, \qquad k < 0 \tag{15}$$

$$C_k = 0, \qquad k = 0 \tag{16}$$

$$C_k = \frac{A_k - iB_k}{2}, \qquad k > 0 \tag{17}$$

$$\omega_k = -\omega_k, \qquad k < 0 \tag{18}$$

Alternatively, we can write C_k in polar form as,

$$C_k = r_k e^{i\phi_k}, \qquad \text{for all } k \tag{19}$$

$$r_k = \left(\frac{A_k^2 + B_k^2}{2}\right)^{1/2} \tag{20}$$

$$\phi_k = \text{atan}\left(\frac{B_k}{A_k}\right), \qquad k < 0 \tag{21}$$

$$\phi_k = \text{atan}\left(-\frac{B_k}{A_k}\right), \qquad k > 0. \tag{22}$$

Finally, putting everything together, (5) in complex form is given as follows,

$$f(t) = \sum_{k=-N}^{N} C_k e^{i2\pi\omega_k t} = \sum_{k=-N}^{N} r_k e^{i\phi_k} e^{i2\pi\omega_k t} = \sum_{k=-N}^{N} r_k e^{i(2\pi\omega_k t + \phi_k)} \tag{23}$$

References

[1] H.J. Weaver. *Applications of Discrete and Continuous Fourier Analysis*. John Wiley and Sons, 1983.