Introduction to the Calculus of Variations

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March 15, 2006

The following gives an introduction to the topic of the *calculus of variations*. In computer vision the calculus of variations has been applied to such problems as estimating optical flow (e.g., (Horn & Schunck, 1981)) and shape from shading (e.g., (Ikeuchi & Horn, 1981)). For an expanded textbook treatment, see (Weinstock, 1974).

The general problem (assuming a single independent variable) is to find the **function** $y(x)$, that makes the following integral stationary (i.e., derivative vanishes):

$$I = \int_{x_1}^{x_2} F(x, y, y')dx, \tag{1}$$

where F is of a known form and x_1 and x_2 are given endpoints; (1) is termed a *functional*, it takes functions as its argument. This is analogous to finding the stationary **points** of a smooth function in calculus.

The starting point is to represent the infinite set of curves passing through the given endpoints that differ from the extremal curve by "small" amounts. These curves, $Y(x)$, are represented by perturbing the extremal curve, $y(x)$, by a function $\eta(x)$ that is zero at x_1 and x_2 and is arbitrary between the endpoints (see Fig. 1):

$$Y(x) = y(x) + \epsilon\eta(x), \tag{2}$$

where ϵ is a scalar parameter. Differentiating (2) with respect to x, yields,

$$Y'(x) = y(x) + \epsilon\eta'(x). \tag{3}$$

It is assumed that $Y(x)$ is C2 continuous (i.e., $y''(x)$ is continuous).

Our new problem is to make $I(\epsilon)$ stationary when $\epsilon = 0$, formally,

$$\frac{dI(\epsilon)}{d\epsilon} = 0 \text{ when } \epsilon = 0, \tag{4}$$

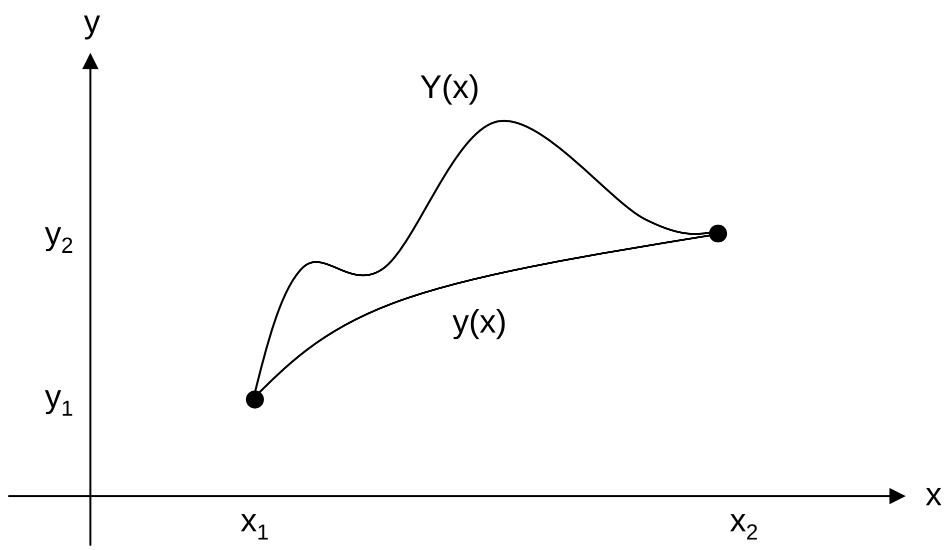

Figure 1: The solution curve, $y(x)$, with an instance of a perturbed curve, $Y(x)$. (x_1, y_1) and (x_2, y_2) denote the given endpoints.

where,

$$I(\epsilon) = \int_{x_1}^{x_2} F(x, Y, Y')dx. \tag{5}$$

Differentiating (5) with respect to ϵ, yields,

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y} \frac{dY}{d\epsilon} + \frac{\partial F}{\partial Y'} \frac{dY'}{d\epsilon} \right) dx. \tag{6}$$

Substituting the derivatives of (2) and (3) wrt ϵ and setting the result to zero at $\epsilon = 0$, yields,

$$\left(\frac{dI(\epsilon)}{d\epsilon} \right)_{\epsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial Y} \eta(x) + \frac{\partial F}{\partial Y'} \eta'(x) \right) dx = 0. \tag{7}$$

Since we are considering $\epsilon = 0$, thus $Y = y$,

$$\left(\frac{dI(\epsilon)}{d\epsilon} \right)_{\epsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \eta'(x) \right) dx = 0. \tag{8}$$

Given the assumption that $y(x)$ is C2 continuous, the second term of the integrand of (8) can be integrated using *integration by parts*:

$$\left(\frac{dI(\epsilon)}{d\epsilon}\right)_{\epsilon=0} = \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx + \left(\frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y'}\eta(x)dx\right) \tag{9}$$

$$= \int_{x_1}^{x_2} \frac{\partial F}{\partial y}\eta(x)dx + \left(0 - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial F}{\partial y'}\eta(x)dx\right); \text{ recall } \eta(x) \text{ is zero at endpoints} \tag{10}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta(x) - \frac{d}{dx}\frac{\partial F}{\partial y'}\eta(x)\right)dx \tag{11}$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta(x)dx. \tag{12}$$

Since $\eta(x)$ is an arbitrary function, this forces the integrand within the brackets to equal zero:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0. \tag{13}$$

Otherwise, we could have selected some function $\eta(x)$ such that the integral would not equal zero, which would violate our assumption that the integral must equal zero. The differential equation given by (13) is known as the *Euler equation* (or *Lagrange equation* in mechanics or *Euler-Lagrange equation*) .

In brief, Horn and Schunck's optical flow approach (Horn & Schunck, 1981) consists of two independent variables, the horizontal and vertical position components x and y, respectively, and two dependent variables, the horizontal and vertical components of velocity given by $u(x, y)$ and $v(x, y)$, respectively. Their problem is formulated as follows,

$$I = \int_{x_1}^{x_2}\int_{y_1}^{y_2} F(x, y, u, v, u', v')dxdy, \tag{14}$$

where

$$F(x, y, u, v, u_x, u_y, v_x, v_y) = [I_x u + I_y v + I_t]^2 + \lambda\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2\right]. \tag{15}$$

The first term of (15) (in square brackets) penalizes deviations from brightness constancy, the second term penalizes deviations from "smoothness" and $\lambda > 0$ is a constant that controls the relative importance between the two terms. For this case the Euler-Lagrange equation can be generalized, as follows,

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u'} - \frac{d}{dy}\frac{\partial F}{\partial u'} = 0 \tag{16}$$

$$\frac{\partial F}{\partial v} - \frac{d}{dx}\frac{\partial F}{\partial v'} - \frac{d}{dy}\frac{\partial F}{\partial v'} = 0. \tag{17}$$

References

Horn, B. & Schunck, B. (1981). Determining optical flow. *Artificial Intelligence*, *17(1-3)*, 185–203.

Ikeuchi, K. & Horn, B. (1981). Numerical shape from shading and occluding boundaries. *Artificial Intelligence*, *17(1-3)*, 141–184.

Weinstock, R. (1974). *Calculus of Variations: With Applications to Physics and Engineering*. Dover.