Overview of Binomial Filters

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This note gives an overview of binomial filters. Binomial filters form a compact rapid approximation of the (discretized) Gaussian. A key application of these filters in computer vision is in the construction of multiscale image/volume representations (e.g., (Burt, 1981; Hummel & Lowe, 1986; Crowley, Riff & Piater, 2003; Lindeberg & Bretzner, 2003)).

The binomial coefficients, given by:

$$\binom{N}{n} = \frac{N!}{(N-n)!N!}, \text{ where } n = 0, \ldots, N, \tag{1}$$

form (when L^1 normalized) a rapid approximation of the normal distribution:

$$G[m] = \frac{1}{\sigma\sqrt{2\pi}} e^{m^2/2\sigma^2}, \tag{2}$$

where $m = n - N/2$, the standard deviation $\sigma = \sqrt{N}/2$, and N (i.e, the order of the binomial filter) denotes the number of cascaded convolutions of the binomial filter, $\begin{bmatrix}1 & 1\end{bmatrix}$, used to generate the binomial filter:

$$B_N = \underbrace{\begin{bmatrix}1 & 1\end{bmatrix} \otimes \begin{bmatrix}1 & 1\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix}1 & 1\end{bmatrix}}_{N \text{ times}}. \tag{3}$$

Alternatively, the binomial filters correspond to the rows of *Pascal's triangle*:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & & \\
 & & & & & & & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

where the sum of the coefficients in a row (i.e., the inverse normalization factor) is 2^N. Two-dimensional binomial filters can be generated by using two one-dimensional binomial filters in a separable fashion, for example:

$$B_2 = (1/4)\begin{bmatrix}1 & 2 & 1\end{bmatrix} \otimes (1/4)\begin{bmatrix}1\\2\\1\end{bmatrix} = (1/16)\begin{bmatrix}1 & 2 & 1\\2 & 4 & 2\\1 & 2 & 1\end{bmatrix}. \tag{4}$$

In one-dimension, the Fourier transform of the binomial filter $B_2 = (1/4)\begin{bmatrix}1 & 2 & 1\end{bmatrix}$ corresponds to a single period of a cosine raised by a constant, in other words, a low-pass filter with no ripples in the stop band (Crowley, Riff & Piater, 2003):

$$\hat{B}_2[u] = \frac{1}{2} + \frac{1}{2}\cos(u). \tag{5}$$

For higher order binomial filters, the Fourier transform is given by the multiplication of B_2's spectra with itself N times: (due to spatial convolution equivalence to frequency multiplication):

$$\hat{B}_N[u] = \left(\frac{1}{2} + \frac{1}{2}\cos(u)\right)^N. \tag{6}$$

Figure 1 illustrates the rapid approximation of the Gaussian by the binomial filters. For further details on binomial filters and empirical evaluations, see (Burt, 1981; Crowley, Riff & Piater, 2003; Jähne, 2005).

References

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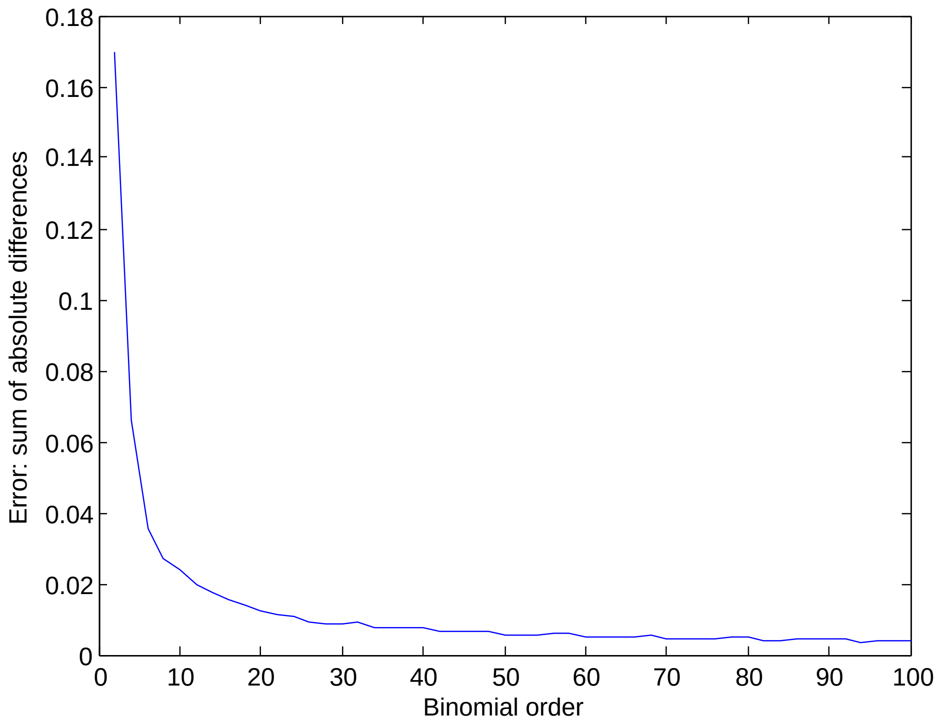

Figure 1: Plot showing the sum of the absolute errors between the Gaussian (truncated at 3σ) and its corresponding binomial filter approximation: $\sum\left|G(x, \sigma = \sqrt{N}/2) - B_N\right|$.