

Overview of Binomial Filters

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This note gives an overview of binomial filters. Binomial filters form a compact rapid *finite impulse response* (FIR) approximation of the (discretized) Gaussian. An appealing implementation feature of these filters is that they do not require multiplications and thus are amenable to hardware implementation (e.g., ([Aubury & Wayne, 1996](#); [Haddad, 1971](#); [van der Wal & Burt, 1992](#))). A key application of these filters in computer vision is in the construction of multiscale image/volume representations (e.g., ([Burt, 1981](#); [Hummel & Lowe, 1986](#); [Crowley, Riff & Piater, 2003](#); [Lindeberg & Bretzner, 2003](#))).

The binomial coefficients, given by:

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}, \text{ where } n = 0, \dots, N, \quad (1)$$

form (when L^1 normalized) a rapid approximation of the Gaussian (normal) distribution:

$$G[m] = \frac{1}{\sigma\sqrt{2\pi}} e^{m^2/2\sigma^2}, \quad (2)$$

where $m = n - N/2$, the standard deviation $\sigma = \sqrt{N}/2$, and N (i.e, the order of the binomial filter) denotes the number of cascaded convolutions (denoted by $\otimes$) of the binomial filter, $\begin{bmatrix} 1 & 1 \end{bmatrix}$, used to generate the binomial filter:

$$B_N = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \end{bmatrix} \otimes \dots \otimes \begin{bmatrix} 1 & 1 \end{bmatrix}}_{N \text{ times}}. \quad (3)$$

Notice that the binomial filters are closed under convolution,

$$B_N = B_{N-M} \otimes B_M. \quad (4)$$

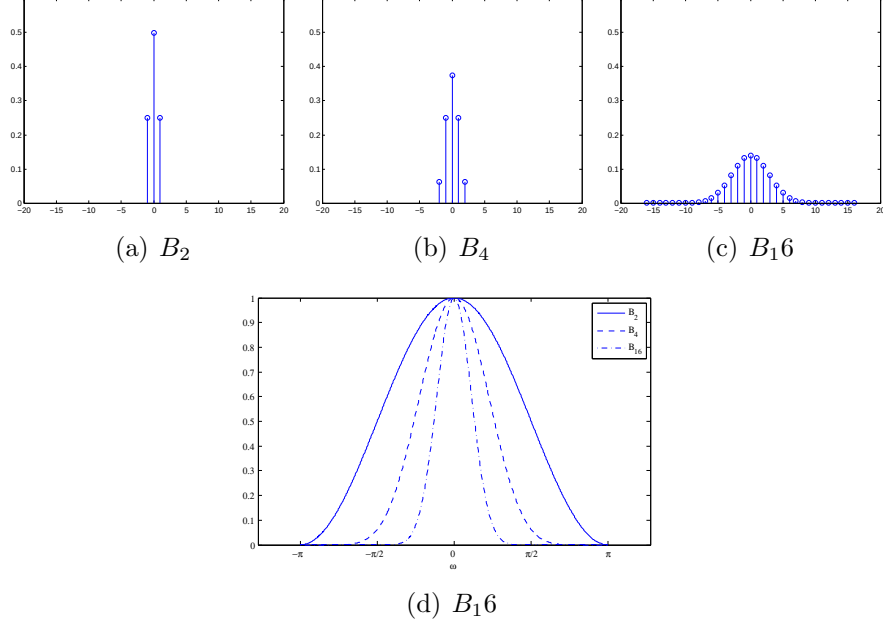


Figure 1: Binomial filters. (a)-(c) depict the impulse responses of several binomial filters. (d) depicts the frequency responses of the binomial filters given in (a)-(c).

Alternatively, the binomial filters correspond to the rows of *Pascal's triangle*:

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & 1 & & 1 & \\
 & & 1 & & 2 & & 1 \\
 & 1 & & 3 & & 3 & & 1 \\
 1 & & 4 & & 6 & & 4 & & 1 \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \vdots
 \end{array}$$

where the sum of the coefficients, given by 2^N , represent the normalization factor (see Fig. 1 (a)-(c)).

Two-dimensional binomial filters can be generated by using two one-dimensional binomial filters in a separable fashion, for example:

$$B_2 = (1/4) \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \otimes (1/4) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = (1/16) \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \quad (5)$$

In one-dimension, the discrete-time Fourier transform (DTFT) of the binomial filter $B_2 = (1/4) \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ corresponds to a single period of a cosine raised by a

constant, in other words, a low-pass filter with no ripples in the stop band:

$$\hat{B}_2[\omega] = \sum_n B_2[n] \cos[\omega n] = \frac{1}{2} + \frac{1}{2} \cos(\omega). \quad (6)$$

For higher order binomial filters, the Fourier transform is given by the multiplication¹ of B_2 's spectra with itself N times (see Fig. 1 (d)):

$$\hat{B}_N[u] = \left(\frac{1}{2} + \frac{1}{2} \cos(u) \right)^N. \quad (7)$$

Figure 2 illustrates the rapid approximation of the Gaussian by the binomial filters. For further details on binomial filters and empirical evaluations, see (Burt, 1981; Haddad, 1971; Nicholson & Davis, 1986; Crowley, Riff & Piater, 2003; Jähne, 2005).

¹This is due to the equivalence of spatial convolution and frequency multiplication.

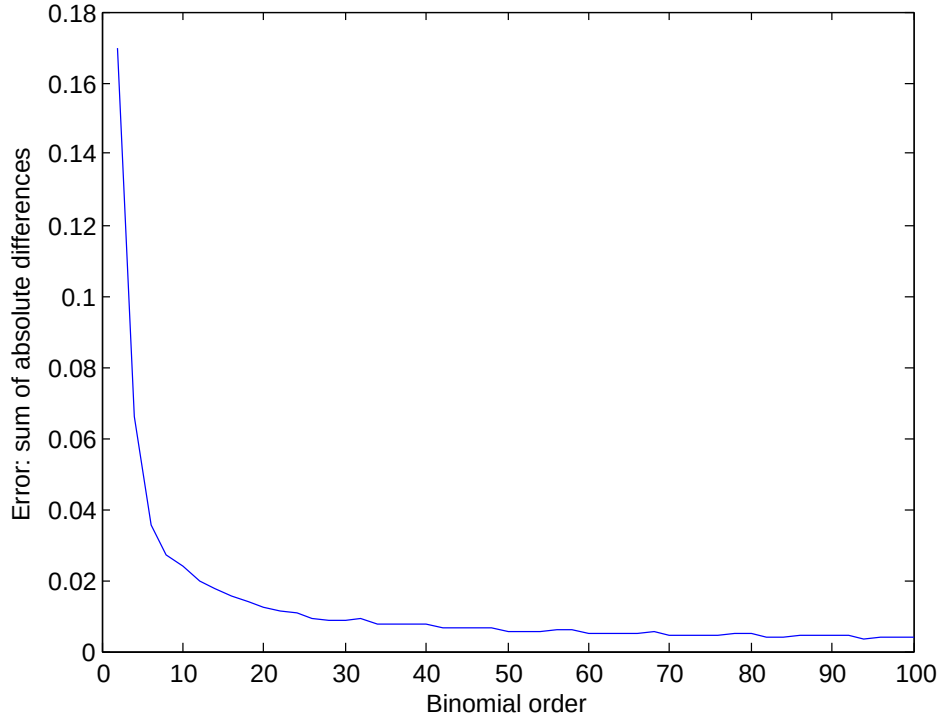


Figure 2: Plot showing the sum of the absolute errors between the Gaussian (truncated at 3σ) and its corresponding binomial filter approximation: $\sum \left| G(x, \sigma = \sqrt{N}/2) - B_N \right|$.

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