Outline of the relationship between the difference-of-Gaussian and Laplacian-of-Gaussian

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The difference-of-Gaussian (DoG) kernel is widely used as an approximation to the scale-normalized Laplacian-of-Gaussian (LoG) kernel (e.g., (Burt & Adelson, 1983; Crowley & Parker, 1984; Lowe, 2004)). In this note the relationship between the difference-of-Gaussian and Laplacian-of-Gaussian image representations is established. This note is adapted from (Marr & Hildreth, 1980; Lowe, 2004).

The DoG image representation, $\text{DoG}(x, y, \sigma)$, is given by:

$$\begin{aligned} \text{DoG}(x, y, \sigma) &= \Big(G(x, y, k\sigma) - G(x, y, \sigma)\Big) * I(x, y) && (1) \\ &= L(x, y, k\sigma) - L(x, y, \sigma), && (2) \end{aligned}$$

where $G(x, y, \sigma)$ represents the Gaussian kernel:

$$G(x, y, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x^2+y^2)}{2\sigma^2}}, \tag{3}$$

and

$$L(x, y, \sigma) = G(x, y, \sigma) * I(x, y). \tag{4}$$

The scale-normalized LoG image representation is given by:

$$\begin{aligned} \text{LoG}(x, y, \sigma) &= \sigma^2 \nabla^2 L(x, y, \sigma) && (5) \\ &= \sigma^2 \Big(L_{xx} + L_{yy}\Big). && (6) \end{aligned}$$

The DoG and LoG can be related through the use of the (heat) diffusion equation,

$$\frac{\partial L}{\partial \sigma} = \sigma \nabla^2 L. \tag{7}$$

The diffusion equation (7) can be approximated as follows,

$$\sigma \nabla^2 L = \frac{\partial L}{\partial \sigma} \tag{8}$$

$$= \lim_{k \to 0} \frac{L(x, y, k\sigma) - L(x, y, \sigma)}{k\sigma - \sigma} \tag{9}$$

$$\approx \frac{L(x, y, k\sigma) - L(x, y, \sigma)}{k\sigma - \sigma} \tag{10}$$

Finally, rearranging (10) yields,

$$(k\sigma - \sigma)\sigma \nabla^2 L \approx L(x, y, k\sigma) - L(x, y, \sigma) \tag{11}$$

$$(k - 1)\sigma^2 \nabla^2 L \approx L(x, y, k\sigma) - L(x, y, \sigma) \tag{12}$$

$$(k - 1)\sigma^2 \mathrm{LoG} \approx \mathrm{DoG}. \tag{13}$$

As can be seen, the DoG approximates the scale normalized LoG up to a (negligible) multiplicative constant, $k - 1$, that is present at all scales.

For an empirical study of the precision of this approximation, see (Crowley & Riff, 2003). Finally, Grabner et al. (Grabner, Grabner & Bischof, 2006) propose a fast approximation of the DoG, the Difference-of-Mean (DoM) representation, that consists of a box filter combined with an *integral image* (c.f. (Viola & Jones, 2004)). Given the integral image, the mean within a rectangular region can be computed in constant time (independent of the size of the region).

References

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