# Embedding into $l_\infty^2$ is Easy, Embedding into $l_\infty^3$ is NP-Complete

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**Abstract**

We give a new algorithm for enumerating all possible embeddings of a metric space (i.e., the distances between every pair within a set of $n$ points) into $R^2$ Cartesian space preserving their $l_\infty$ (or $l_1$) metric distances. Its expected time is $\mathcal{O}(n^2 \log^2 n)$ (i.e. within a poly-log of the size of the input) beating the previous $\mathcal{O}(n^3)$ algorithm. In contrast, we prove that detecting $l_\infty^3$ embeddings is NP-complete. The problem is also NP-complete within $l_1^2$ or $l_\infty^2$ with the added constraint that the locations of two of the points are given or alternatively that the two dimension are curved into a 3-dimensional sphere. We also refute a compaction theorem by giving a metric space that cannot be embedded in $l_\infty^3$, however, can be if any single point is removed.

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# 1 Introduction

An $n$-point metric space consists of the distances $d(P, P')$ between every pair of points $P$ and $P'$ within a set of $n$ points $\chi$. We say that it can be *isometrically embedded* into $l_q^k$ (the Cartesian space $R^k$ endowed with the $l_q$-distance) if there exists a distance-preserving mapping $\varphi : \chi \rightarrow R^k$, i.e.

$$\forall P, P' \in \chi,\ d(P, P') = ||\varphi(P) - \varphi(P')|| = \left( \sum_{i=1}^{k} |P_{x_i} - P'_{x_i}|^q \right)^{1/q}.$$

Finding such an embedding, if possible, is not difficult for $1 < q < \infty$ because the location in $R^k$ of a point is completely determined by knowing its $l_q$ distance to $k+1$ linearly independent locations. The same is not true for $q = 1$ with $||\varphi(P) - \varphi(Q)|| = \sum_{i=1}^{k} |P_{x_i} - Q_{x_i}|$ or for $q = \infty$ with $||\varphi(P) - \varphi(Q)|| = \max_{i=1}^{k} |P_{x_i} - Q_{x_i}|$, because with these metrics, there may be an infinite number of locations with the same distances to any given finite set of points.

Malitz and Malitz in 1992 [14] gave a $\mathcal{O}(n^3)$ algorithm for embedding an arbitrary $n$-point metric space into $l_1^2$ or into $l_\infty^2$. Alternative algorithms and information on this can be found in [10, 4, 6, 11]. Our first result is a new polynomial time algorithm which does the same with expected $\mathcal{O}(n^2 \log^2 n)$ time (i.e. within a poly-log of the size of the input). More over this algorithm can enumerate all such embeddings.

Avis and Deza [3] show that deciding if a finite metric space embeds into $l_1^k$ embeddings for arbitrary $k$ is NP-complete. It is conjectured that embedding into $l_1^k$ can be done in $n^{\mathcal{O}(k)}$ time and hence polynomial for a fixed $k$. In contrast, we prove that detecting $l_\infty^3$ embeddings is NP-complete. The problem is also NP-complete for $l_1^2$ or $l_\infty^2$ with the added constraint that the locations of two of the points are given or alternatively that the two dimensions are curved into a 3-dimensional sphere.

For a more combinatorial version of the problem, define $c_q(k)$ to be the smallest integer, if it exists, such that a metric space embeds into $l_q^k$ if and only if all subspaces with at most $c_q(k)$ points embed. Note that this automatically gives a $\mathcal{O}(n^{c_q(k)})$ time algorithm for detecting $l_q^k$ embeddings simply by checking all $\binom{n}{c_q(k)}$ subsets of the points of cardinality $c_q(k)$. Merger in 1928, [16, 9], proved a compactness theorem stating that $c_2(k) = k + 3$, A similar theorem is likely also true for $1 < q < \infty$. Malitz and Malitz, [14], proved that for $l_1^2$ and $l_\infty^2$ only 11 points are needed. In [14, 17], they prove that the number of points for $l_1^k$ is at least $2k + 2$, which they conjectured is tight. Though [5] confirmed that $c_1(2) = 6$, [5, 7] disproved this conjecture by showing that $c_1(3) \geq 10$, and $c_1(k) \geq k^2$. This paper focuses on the $l_\infty$ metric. Besides $c_\infty(2) = 6$, little was known. It is asked as problem 3.13 from the list "Open problems on embeddings of finite metric spaces" edited by J. Matousek [15]. Our proving that detecting $l_\infty^3$ embeddings is NP-complete proves that unless $P = NP$ there is no poly-time algorithm for this and hence $c_\infty(3)$ cannot be a constant. This paper goes on by giving for any $n \geq 24$ a metric space that cannot be embedded in $l_\infty^3$, however, can be if any single point is removed, effectively showing that $c_\infty(3)$ is unbounded.

The model for the algorithm is as follows. Because the input consists of $\mathcal{O}(n^2)$ real values for the distances between the $n$-points and because an embedding consists of real valued coordinates for the points, the machine must be able to manipulate real numbers and perform simple arithmetic operations on them in constant time. Bandelt and Chepoi [6] outline how to derive all embeddings in $l_1$-space from a single one. The embeddings to be "enumerated" fall into at most $\mathcal{O}(n)$ different classes. Within one class of embeddings, the points fall into components, each of which is embedded in a fixed way relative to itself, but which can be transposed and flipped relative to the other components. There being an infinite number of these transpositions and an exponential number of these ways of flipping, we clearly do not have time to enumerate each, but we are able to describe the range of motion in $\mathcal{O}(n)$ space so that all $\mathcal{O}(n)$ of these classes can be outputted in the required $\mathcal{O}(n^2 \log^2 n)$ time. We will give the embedding algorithm for $l_\infty^2$ and not for $l_1^2$, because of the well known translation between these two spaces.

# 2 Algorithm for Embedding into $l_1^2$ or $l_\infty^2$

The main goal of this section is to prove the following result.

**Theorem 1** *Given an n-point metric space $(\chi, d)$, one can enumerate all its possible isometric embeddings into $l_\infty^2$ or $l_1^2$ in a total of $\mathcal{O}\left(n^2 \log^2 n\right)$ time.*

**Proof of Theorem 1:** Let $U$ and $V$ be a diametral pair of $(\chi, d)$ and let $\widehat{y} = d(U, V)$. Modulo translation and change in dimensions, we can assume without loss of generality that the points $U$ and $V$ are embedded at locations $\varphi(U) = \langle 0, 0\rangle$ and $\varphi(V) = \langle \widehat{x}, \widehat{y}\rangle$, for some unknown value $\widehat{x} \in [0, \widehat{y}]$. Since, $||\varphi(U) - \varphi(V)|| = \max(\widehat{x}, \widehat{y}) = \widehat{y}$, we say that this distance $d(U, V)$ is *manifested* in the $Y$ dimension. Given any point $P \in \chi$ different from $U$ and $V$, we will look for the location $\varphi(P) = \langle P_x, P_y\rangle$ at which to embed it. Set $u_P = d(U, P)$ and $v_P = d(P, V)$ (or simply $u$ and $v$ when $P$ is understood). Define $\mathcal{S}$ to be the set of all points $Q$ of $\chi$ which are *between* $U$ and $V$, i.e. such that $d(U, V) = d(U, Q) + d(Q, V) = u_Q + v_Q$. A key issue is how the difference $v_P - u_P$ compares to $\widehat{x}$. To help us determine this, define the set of values $\Delta = \{|v_P - u_P| \mid P \notin \mathcal{S}\}$. Sort these values and let $x_0 = 0$, $x_i$ be the $i^{th}$ distinct value in $\Delta$, and $x_{n+1} = \infty$. Separately, for each $i$, the algorithm will enumerate all possible embeddings of the points into $l_\infty^2$ in which the unknown value $\widehat{x}$ is equal to $x_i$ and then those for which $\widehat{x}$ is strictly within the interval $(x_i, x_{i+1})$. From here on, let us restrict our attention to $\widehat{x}$ being in one such interval. Note that this restriction allows us to compare $\widehat{x}$ to $|v_P - u_P|$ for each point $P$. In fact, the next step of the algorithm is to classify each point $P$ according to this comparison. Section 2.1 uses this classification to narrow where each point can be embedded to one of two regions and then to narrow each point down to only one of these two regions. Section 2.2 then manages either to fix the $X$-dimension of every point or to fix the $Y$ dimension of every point. From here, it partitions the points into components, each of which is embedded in a fixed way relative to itself, but which can be transposed and flipped relative to the other components. Finally, Section 2.3 describes how in a total of $\mathcal{O}(n^2 \log^2 n)$ time, the above tasks can be repeated for each of the $\mathcal{O}(n)$ different intervals that $\widehat{x}$ is restricted to.

## 2.1 Classifying The Points

Consider some $i \in [0, n]$ and either fix the unknown value $\widehat{x}$ to equal $x_i$ or to be strictly within the interval $(x_i, x_{i+1})$. Classify each point $P$ into the six categories as shown in the first column of the following table. Whether $P \in \mathcal{S}$ is easy to determine. For the remaining points, note that $|v - u|$, which is short for $|v_P - u_P| = |d(P, V) - d(P, U)|$, is in the set $\Delta$. Because we have fixed how $\widehat{x}$ compares to the values in $\Delta$, we can determine whether or not $v - u > \widehat{x}$. If so, classify $P \in \mathcal{A}$. Similarly, for the remaining classifications. Depending on the classification of $P$, the table goes on to define two regions of locations $R^1(P)$ and $R^2(P)$ within which the point must be embedded. For example, if point $P$ is classified as $\mathcal{S}$, then $R^1(P) = \langle ?, u\rangle$. This states that $P$ must be embedded at $\langle P_x, P_y\rangle$ where $P_y = u$ and the $X$-coordinate $P_x$ is not determined. Figure 1.a goes on to specify that $P$ must be embedded in the region of locations labeled $L'_{\mathcal{S}}$. As such, it specifies the range of values that $P_x$ can have to be within the dotted line labeled $\langle ?, u\rangle$ in the figure. Similarly, if point $P$ is classified as $\mathcal{A}$, $P$ must be embedded either at the location $R^1(P) = \langle -u, \widehat{y} - v\rangle$ or location $R^2(P) = \langle u, \widehat{y} - v\rangle$. These locations fall within the regions $L'_{\mathcal{A}}$ and $L''_{\mathcal{A}}$ in Figure 1.a.

| | | |
|---|---|---|
| $\mathcal{S} = \{P \mid u + v = \widehat{y}\}$ | $R^1(P) = \langle ?, u\rangle = \langle ?, \widehat{y} - v\rangle$ | |
| $\mathcal{A} = \{P \notin \mathcal{S} \mid v - u > \widehat{x}\}$ | $R^1(P) = \langle -u, \widehat{y} - v\rangle$ | $R^2(P) = \langle u, \widehat{y} - v\rangle$ |
| $\mathcal{B} = \{P \notin \mathcal{S} \mid v - u = \widehat{x}\}$ | $R^1(P) = \langle -u, ?\rangle = \langle \widehat{x} - v, ?\rangle$ | $R^2(P) = \langle u, \widehat{y} - v\rangle$ |
| $\mathcal{C} = \{P \notin \mathcal{S} \mid \lvert v - u\rvert < \widehat{x}\}$ | $R^1(P) = \langle \widehat{x} - v, u\rangle$ | $R^2(P) = \langle u, \widehat{y} - v\rangle$ |
| $\widehat{\mathcal{B}} = \{P \notin \mathcal{S} \mid u - v = \widehat{x}\}$ | $R^1(P) = \langle \widehat{x} - v, u\rangle$ | $R^2(P) = \langle u, ?\rangle = \langle \widehat{x} + v, ?\rangle$ |
| $\widehat{\mathcal{A}} = \{P \notin \mathcal{S} \mid u - v > \widehat{x}\}$ | $R^1(P) = \langle \widehat{x} - v, u\rangle$ | $R^2(P) = \langle \widehat{x} + v, u\rangle$ |

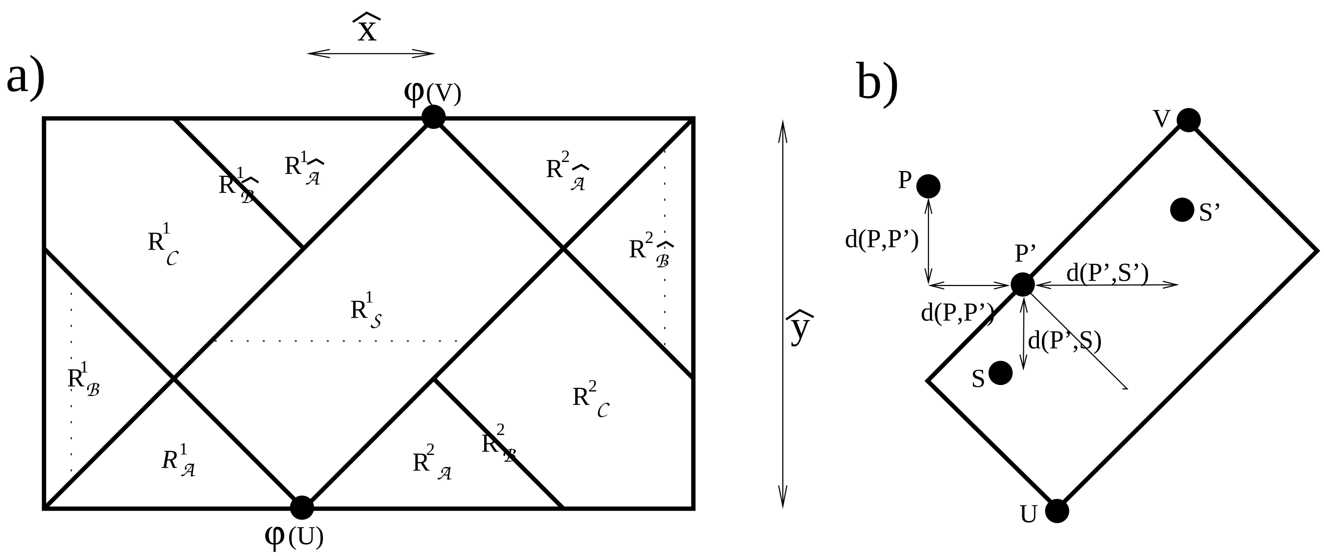


Figure 1: The space $l^2_\infty$ is partitioned into the areas according to how points are embedded into them. For example, if point $P$ is classified as $\mathcal{A}$, $P$ must be embedded either at the location $R^1(P) = \langle -u, \widehat{y} - v\rangle$ or location $R^2(P) = \langle u, \widehat{y} - v\rangle$. These locations fall within the regions labeled $L'_{\mathcal{A}}$ and $L''_{\mathcal{A}}$ respectively.

Alternatively, these ideas can be expressed using the well known notion of a gated set. A subset $\mathcal{S}$ of a metric space is called *gated* if every point $P$ outside $\mathcal{S}$ contains a (necessarily unique) point $P' \in \mathcal{S}$ (the gate for $P$ in $\mathcal{S}$) such that for every point $S \in \mathcal{S}$, $d(P, S) = d(P, P') + d(P', S)$. The set $\mathcal{S}$ with corners $U$ and $V$ is a gated set of $l^2_\infty$. See Figure 1.b. Our classification of the points $P$ corresponds to which edge or corner of $\mathcal{S}$ the gate $P'$ of $P$ is on. We do this classification based on the value $v_P - u_P$. Alternatively, we could consider the values $\alpha_P = d(U, P') = \frac{1}{2}[d(U, P) + d(U, V) - d(P, V)] = \frac{1}{2}[u_P + \widehat{y} - v_P]$ and $\beta_P = d(V, P') = \frac{1}{2}[d(V, P) + d(U, V) - d(P, U)] = \frac{1}{2}[v_P + \widehat{y} - u_P]$. If the position of $P$ is unknown, from the knowledge of $u_p$ and $v_p$ we infer that there exist only two possible locations of the gate $P'$ of $P$. As well, if the location of this gate is known and is not a corner of $\mathcal{S}$, then we can precisely deduce the location of $P$. In addition, we can also tell whether or not the gates of two points $P$ and $W$ belong to the same (open) side of $\mathcal{S}$. This can be used to place $P$ once $W$ has been placed.

**Lemma 1** *If $\widehat{x}$ is restricted as stated, then point $P$ must be embedded either within the defined region $R^1(P)$ or $R^2(P)$. For a fixed i, this classification can be done easily in time $\mathcal{O}(n)$.*

**Proof of Lemma 1:** Consider some point $P$. It must be embedded within the rectangle $-(\widehat{y}-\widehat{x}) \leq P_x \leq \widehat{y}$ and $0 \leq P_y \leq \widehat{y}$ otherwise the distance from $P$ to either $U$ or $V$ would be greater than $\widehat{y}$, contradicting the choice of $U$ and $V$. See Figure 1.a.

The input states that the distance between $P$ and point $U$ is $d(U,P)$, which we are denoting $u$. Since $\varphi(U) = \langle 0,0\rangle$, we obtain $u = ||\varphi(P) - \varphi(U)|| = \max(|P_x - 0|, |P_y - 0|)$. This places $P$ within the half square of radius $u$ around $U$, namely $\langle P_x, P_y\rangle \in \{\langle -u, ?\rangle, \langle ?, u\rangle, \langle u, ?\rangle\}$. Note that these three options correspond to the point being embedded left, above, or right of the two $45^o$ lines emanating up from location $\varphi(U)$. We have been also been told that the distance from $P$ to $V$ is $v$, which places it within the half square of radius $v$ around $V$, namely either $\langle P_x, P_y\rangle \in \{\langle \widehat{x} - v, ?\rangle, \langle ?, \widehat{y} - v\rangle, \langle \widehat{x} + v, ?\rangle\}$, which correspond to being embedded left, below, or right of the two $45^o$ lines emanating down from locations $\varphi(V)$. $P$ must be embedded in the intersection of these two half squares. See Figure 2.

If $u + v = \widehat{y}$, then these two half squares intersect only along their edge above $U$ and below $V$ as shown in Figure 2:$\mathcal{S}$. We argue in this case that the distances $u_P$ and $v_P$ are both manifested in the $y$-dimension and hence its embedding in the $y$-dimension is $P_y = u = \widehat{y} - v$. If $P_y$ were bigger than this, then the distance from $P$ to $U$ would be bigger than $u$ and if it were smaller, then the distance from $P$ to $V$ would be bigger than $v$.

If $u + v < \widehat{y}$, then the two rectangles around $U$ and $V$ do not intersect and by the triangle inequality no embedding is possible.

When $u + v > \widehat{y}$, how the two squares intersect depends on the ordering of the sides of the squares. Let $LU = -u$, $RU = u$, $LV = \widehat{x} - v$ and $RV = \widehat{x} + v$ denote the $X$-coordinate of the right and left sides of the squares around $U$ and $V$.

If $P \in \mathcal{A}$, then $v - u > \widehat{x} \geq 0$, giving $LV = \widehat{x} - v < -u = LU < RU = u < -\widehat{x} + v < \widehat{x} + v = RV$. See Figure 2:$\mathcal{A}$. Therefore, the intersection is either on the left or the right sides of $U$'s square and on the bottom of $V$'s. In the first case, $P$ is embedded at $\langle P_x, P_y\rangle = \langle -u, \widehat{y} - v\rangle$, which is denoted by $R^1(P)$. In the second, at $\langle u, \widehat{y} - v\rangle$, denoted $R^2(P)$.

If $P \in \mathcal{B}$, then $v - u = \widehat{x}$, giving $LV = \widehat{x} - v = -u = LU < RU = u = -\widehat{x} + v < \widehat{x} + v = RV$. See Figure 2:$\mathcal{B}$. Therefore, the intersection is either left of both $U$ and $V$ or right of $U$ and bottom of $V$. In the first case, $P$ is embedded at $\langle P_x, P_y\rangle = \langle -u, ?\rangle = \langle \widehat{x} - v, ?\rangle$, denoted $R^1(P)$. The second case is the same, as before.

If $P \in \mathcal{C}$, then $v - u < \widehat{x}$, giving $LU = -u < \widehat{x} - v = LV$ and $RU = u < \widehat{x} + v = RV$. We have $\widehat{x} \leq \widehat{y}$, by the fact that $d(U,V) = \widehat{y}$ and not $\widehat{x}$ and we have $\widehat{y} < u + v$, by $P \notin \mathcal{S}$. Hence, $LV = \widehat{x} - v \leq \widehat{y} - v < u = RU$. See Figure 2:$\mathcal{C}$. Therefore, the intersection is either top of $U$ and right of $V$ or left of $U$ and bottom of $V$. In the first case, $P$ is embedded at $\langle \widehat{x} - v, u\rangle$, denoted $R^1(P)$. The second case is the same as before.

The other cases are the same except for the roles of $U$ and $V$ switched. ■

We have narrowed the embedding of each point down to one of two specified regions. We will now describe how for each point in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, we will narrow this down to one region. ($\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$, and $\mathcal{C}$ are done in a symmetric way.) If there are no such points, then there is nothing to do in this

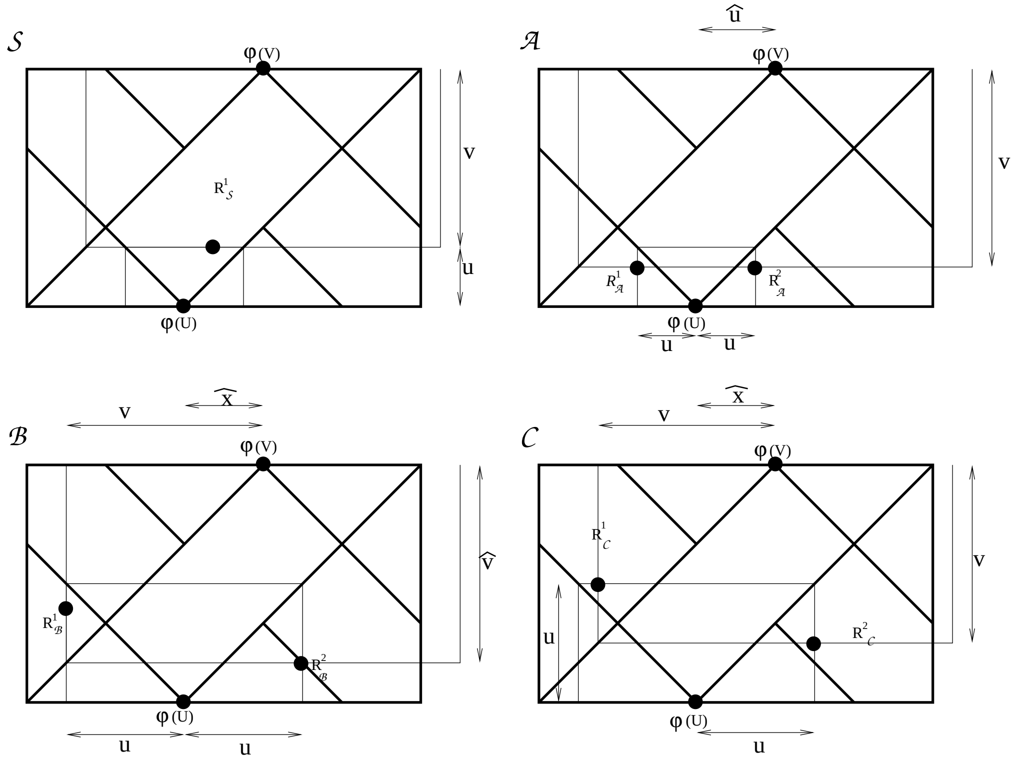


Figure 2: For each classification of points $P$, it is shown how the square around $U$ and that around $V$ intersect at two regions.

task. Otherwise, let $W$ be one of these points $P$ that maximizes $v_P - u_P = d(P,V) - d(P,U)$. This point $W$ is either embedded at $R^1(W)$ or $R^2(W)$. The algorithm will branch twice trying each of these possibilities.

To make it concrete, let $j, k \in \{1, 2\}$. Suppose we are trying to embed $W$ in $R^j(W)$. Consider some point $P$ in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ and let us try to embedded it within $R^k(P)$. Also, if our current case allows $\widehat{x}$ to vary within some interval $(x_i, x_{i+1})$, then for the moment fix its value. Even with these restrictions, there may be a lot of possibilities as to the locations $\varphi(W)$ and $\varphi(P)$ for $W$ and $P$. Denote the set of possible distances between these embeddings to be $D^{j,k}(P) = \{||\varphi(W) - \varphi(P)|| \mid \varphi(W) \in R^j(W) \text{ and } \varphi(P) \in R^k(P)\}$. Clearly, such an embedding of $W$ and $P$ is impossible if the required distance $d(W,P)$ is not in this set. Lemma 2 proves that the sets of distances $D^{j,1}(P)$ is disjoint from the set $D^{j,2}(P)$. Hence, $d(W,P)$ cannot be in both of them, giving that the location of $P$ has been narrowed down to at most one region. If in this process $P$ is narrowed down to neither region, then the algorithm reports that there are no embeddings consistent with these choices made so far. Then the next embeddings of $W$ or interval for $\widehat{x}$ is tried. Now return to the fact that $\widehat{x}$ may vary within the interval $(x_i, x_{i+1})$. Lemma 2 will go on to prove that which of $R^1(P)$ or $R^2(P)$ the above method chooses does not depend on the value of $\widehat{x}$ within this range.

**Lemma 2** *For $j \in \{1, 2\}$ and $P \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, the sets of possible distances $||R^j(W) - R^1(P)||$ is disjoint from the set $||R^j(W) - R^2(P)||$. More over, the choice of $R^1(P)$ or $R^2(P)$ does not depend on the value of $\widehat{x}$ within $(x_i, x_{i+1})$. The case with $P \in \widehat{\mathcal{A}} \cup \widehat{\mathcal{B}} \cup \mathcal{C}$ is symmetric.*

**Proof of Lemma 2:** We distinguish four cases.

**Case 1: $W$ and $P$ are both in $\mathcal{A} \cup \mathcal{B}$ and $W$ is embedded in $R^1(W)$.** Let $\varphi^1(P) \in R^1(P)$, $\varphi^2(P) \in R^2(P)$, and $\varphi(W) \in R^1(W)$. It is sufficient to prove that $||\varphi(W) - \varphi^1(P)|| < ||\varphi(W) - \varphi^1(P)||$. Because $P \in \mathcal{A} \cup \mathcal{B}$, $\varphi^1(P) = \left\langle -P_x, P'_y \right\rangle$ and $\varphi^2(P) = \left\langle +P_x, P''_y \right\rangle$ with $P_x = u(P) > 0$ (or else $P = U$), $0 \leq P'_y \leq P_x$, and $0 \leq P''_y \leq P_x$ (because $u(P)$ is manifested in the $X$-dimension). Similarly, because $W \in \mathcal{A} \cup \mathcal{B}$, it is located at $\varphi(W) = \langle -W_x, W_y \rangle$ with $0 \leq W_y \leq W_x$. Therefore, $|W_y - P'_y| \leq W_y + P'_y \leq W_x + P_x = |(-W_x) - (P_x)|$. The only way to have equality here is if $W_y = W_x$, $P'_y = P_x$ and either $W_y$ or $P'_y$ is zero, in which case either $W$ or $P$ is equal to $U$, which we assume is not the case. Hence, $|W_y - P'_y| < |(-W_x) - (P_x)|$. Similarly, $|W_y - P''_y| < |(-W_x) - (P_x)|$. We also have that $|(-W_x) - (-P_x)| < |(-W_x) - (P_x)|$. Therefore, $||\varphi(W) - \varphi^1(P)|| = \max(|(-W_x) - (-P_x)|, |W_y - P'_y|) < |(-W_x) - (P_x)| = \max(|(-W_x) - (P_x)|, |W_y - P''_y|) = ||\varphi(W) - \varphi^2(P)||$. If the value of $\widehat{x}$ changes within $(x_i, x_{i+1})$ then the points do not change categories and the locations considered do not move. (Note that if a point is in $\mathcal{B}$, the $\widehat{x}$ is fixed to some $x_i$.)

**Case 2: $W$ and $P$ are both in $\mathcal{A} \cup \mathcal{B}$ and $W$ is embedded in $R^2(W)$.** This case is similar to the last except for flipping around the $X$-axis.

**Case 3: $W \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ and $P \in \mathcal{C}$.** For $P \in \mathcal{C}$, the regions $\varphi^1 = R^1(P) = \langle \widehat{x} - v, u \rangle$ and $\varphi^2 = R^2(P) = \langle u, \widehat{y} - v \rangle$ consist of a single point (fix the value of $\widehat{x}$ for the moment). Let $\mathcal{Q}$ be the set of the locations $Q$ that are at equal distance from locations $\varphi^1$ and $\varphi^2$. The method proves that these locations are all contained in the region $\varphi^1_{\mathcal{S}}$, i.e. above or on $U$'s upper $45^o$ lines and below or $V$'s lower $45^o$ lines. See Figure 3:b. Because $W \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, all the locations in $R^1(W)$ are to the left of $\varphi^1_{\mathcal{S}}$. Hence the distance from $R^1(P)$ to $R^1(W)$ is different then the distance from $R^2(P)$ to $R^1(W)$. Similarly, all the locations in $R^2(W)$ are to the right of $\varphi^1_{\mathcal{S}}$ and hence the distance from $R^1(P)$ to $R^2(W)$ is different then the distance from $R^2(P)$ to $R^2(W)$. Again, if the value of $\widehat{x}$ changes within $(x_i, x_{i+1})$, then the points do not change categories and the fact that $R^1(W)$ and $R^2(W)$ are on opposite sides of $\mathcal{Q}$ does not change.

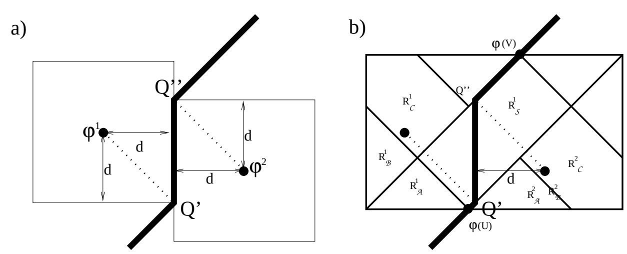


Figure 3: The bold line in (a) gives the set of points $Q$ whose distances to $C'$ is equal to its distance to $C''$. (b) demonstrates how this region lies within completely in the region $\mathcal{S}$.

We will leave it up to the reader to see that the equal distance loci set $\mathcal{Q} = \{Q \in R^2 \mid ||\varphi^1 - Q|| = ||\varphi^2 - Q||\}$ consists, as is marked in Figure 3:a, of the line segment from $Q'$ to $Q''$, the $45^o$ line down and to the left of $Q'$, and the $45^o$ line up and to the right of $Q''$.

To verify the picture, we first prove that the distance between locations $\varphi^1$ and $\varphi^2$ is manifested as shown in the $X$-dimension. $\varphi^1$ is left of $\varphi^2$, because $\varphi^1_x = \widehat{x} - v \leq \widehat{y} - v < (u+v) - v = u = \varphi^2_x$; $\varphi^1$ is above of $\varphi^2$, because $\varphi^1_y = u = (u+v) - v \geq \widehat{y} - v = \varphi^2_y$; and the $X$-distance is larger, because $\Delta x = u - (\widehat{x} - v) \geq u - (\widehat{y} - v) = \Delta y$.

We compute $d = ||\varphi^2 - \varphi^1||/2 = (u - (\widehat{x} - v))/2$. $Q'$, which is the point right $d$ and down $d$ from $\varphi^1$, is $\left\langle \varphi^1_x + d, \varphi^1_y - d \right\rangle = \langle (\widehat{x} - v + u)/2, (\widehat{x} - v + u)/2 \rangle$, and $Q''$, which is the point left $d$ and up $d$ from $\varphi^2$, is $\left\langle \varphi^2_x - d, \varphi^2_y + d \right\rangle = \langle \widehat{x} - (\widehat{x} - v + u)/2, \widehat{y} - (\widehat{x} - v + u)/2 \rangle$. Note that $Q'$ is on $U$'s upper right $45^o$ line and $Q''$ is on $V$'s lower left $45^o$ line. It follows that the locations $Q$ that are equal distance from locations $\varphi^1$ and $\varphi^2$ are all contained in the region $\mathcal{S}$, i.e. above or on $U$'s upper $45^o$ lines and below or $V$'s lower $45^o$ lines, and hence $W$ is not equal distant from them.

**Case 4: $W \in \mathcal{C}$ and $P \in \mathcal{A} \cup \mathcal{B}$.** This case is impossible. Because $W$ is one of the points that maximizes $v_P - u_P$, if $W$ is not in $\mathcal{A}$ or $\mathcal{B}$ then these classes are empty. ■

Every point is now completely narrowed down to one region $R^1(P)$ or $R^2(P)$. The points in $\mathcal{B}'$ and $\widehat{\mathcal{B}}''$ are not fixed in the $Y$ dimension, those in $\mathcal{S}$ are not fixed in the $X$-dimension, and many are not fixed in the $X$-dimension because value of $\widehat{x}$ within $(x_i, x_{i+1})$ is unknown. The next step is to either fix the $X$-dimension of every point or fix the $Y$ dimension of every point. There are two cases.

In the first case (see Figure 4.a), there are no points $B \in \mathcal{B}' \cup \widehat{\mathcal{B}}''$. Hence, every point is fix in the $Y$ dimension. The algorithm at this point, completely relaxes the restriction on $\widehat{x}$. We are now in the situation that every point is fix in the $Y$ dimension, but their $X$-coordinate $P_x$ is unknown.

In the second case (see Figure 4.b), there is a point $B \in \mathcal{B}' \cup \widehat{\mathcal{B}}''$. If $B \in \mathcal{B}'$, then $\widehat{x}$ is known to be equal to $v_B - u_B$ (in fact, we started this attempt at embedding by narrowing $\widehat{x}$ down to this single value $x_i$). More over, the distance from such a $B$ to any point $P \in \mathcal{S}$ is determined in the $X$-dimension. See Figure 1.a to see that the $X$-distance between $B$ and $P$ is more than the $Y$ distance. We do not know the $Y$ coordinate of $B$, but we do know it's $X$-coordinate, $B_x$. From this we know that $P_x = B_x + d(B, P)$. This fixes the location of all the points in $\mathcal{S}$. The same thing can be done if there is a point in $\widehat{\mathcal{B}}''$. Either way, every point is fix in the $X$-dimension, but those in $\mathcal{B}' \cup \widehat{\mathcal{B}}''$ are free in the $Y$ dimension.

In the next section, we will be focusing on the Figure 4.a case, but the other is similar.

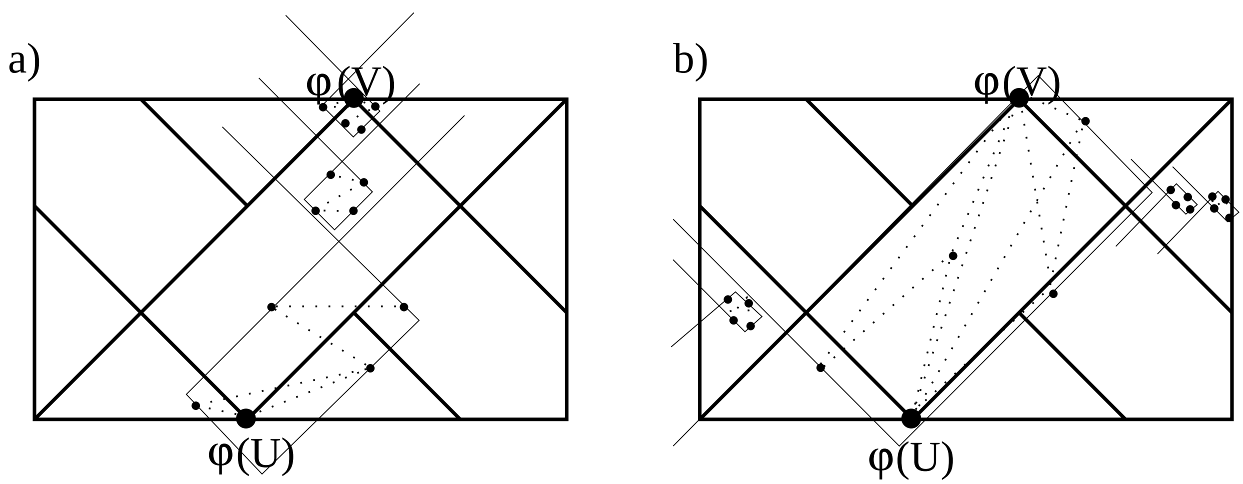


Figure 4: Two examples of embeddings. In (a) one, the components move within the $X$-dimension and in (b) within the $Y$.

## 2.2 Embedding with all but one Dimension Fixed

Suppose we are given a metric space (i.e., the distances between every pair of points) and we are given a partial embedding. The coordinates for one of the dimensions, say $X$, may be unknown, yet the coordinates in the other dimensions are fixed. The question for this section is whether or not this metric space can be embedded in a way consistent with the partial embedding. We will show that there is potentially an exponential number of ways in which components of points might flip and potentially an infinite number of ways for these components to translate within $X$-coordinate. Clearly these are not all enumerated, but this section will described how they are all characterized. In order to be more applicable, we will initially consider the possibility of more than two dimensions.

Consider the graph on the points of the metric space with an edge $\{P, Q\}$ if $d(P, Q) > ||P-Q||_{\vec{y}}$, where $d(P, Q)$ is the distance required by the input and $||P - Q||_{\vec{y}} = \max(|P_y - Q_y|, \ldots)$ is the distance determined by the dimensions fixed so far. Such distances $d(P, Q)$ must be manifested in the $X$-dimension. Partition this graph into connected components. (In Theorem 3, there is a single component forming a single path.)

**Lemma 3** *Let $\Phi$ be one of the connected components. Modulo translating and flipping $\Phi$ as a unit along the $X$-dimension, the locations of all the points in the component are fixed. For each point $P \in \Phi$, the algorithm returns $d_x(\Phi, P)$, which is the relative $X$-location of $P$ with respect to some designated spot within this rigid $\Phi$. Then if someone else provides $\Phi_x$, which we use to denote the actual embedded $X$-coordinate of the designated spot, and $flip(\Phi) \in \{1, -1\}$, which we use to denote whether or not $\Phi$ is flipped, then the actual $X$-coordinate of point $P$ will be $P_x = \Phi_x + flip(\Phi) \cdot d_x(\Phi, P)$.*

**Proof of Lemma 3:** Consider this either to be a proof by induction on the number of nodes in the component or an algorithm with recursion. There are a few cases. For a single point $P$, the designated spot will clearly be this point itself, giving $d_x(\Phi, P) = 0$.

If the component consists of a single edge $\{P, Q\}$, then because the distance $d(P, Q)$ must be manifested in the $X$-dimension, $Q_x$ is must be either to $P_x + d(P, Q)$ or $P_x - d(P, Q)$. Setting $d_x(\Phi, Q) = d(P, Q)$ gives $P_x = \Phi_x + flip(\Phi) \cdot 0$, $Q_x = \Phi_x + flip(\Phi) \cdot d(P, Q)$, and $|Q_x - P_x| = |flip(\Phi) \cdot (d(P, Q) - 0)| = d(P, Q)$ as required.

Now consider a component of any size. Let $R$ be a leaf of some spanning tree, let $Q$ be one its neighbors, and $P$ be one of $Q$'s neighbors. By induction/recursion, the component with $R$ removed is rigid. If there is an edge $\{P, R\}$ (see Figure 5.a), then the two constraints $R_x = P_x \pm d(P, R)$ and $R_x = Q_x \pm d(Q, R)$ fix the embedding of $R$ within the component. (Note $P_x$ and $Q_x$ must be different or else the distance between them can't be manifested in the $X$-dimension.)

If there is not an edge $\{P, R\}$ (see Figure 5.b), then we claim that $R_x$ must be fixed on the same side of $Q_x$ that $P_x$ is. This gives $d_x(\Phi, R) = d_x(\Phi, Q) + d(Q, R)$ if $d_x(\Phi, P) > d_x(\Phi, Q)$ and $d_x(\Phi, R) = d_x(\Phi, Q) - d(Q, R)$ if $d_x(\Phi, P) < d_x(\Phi, Q)$. The proof that this works supposes by way of contradiction that $P_x$ is set to $Q_y - d(P, Q)$ and $R_x$ is set to $Q_y + d(Q, R)$. This gives an embedded distance from $P$ to $R$ of at least $d(P, Q) + d(Q, R)$. Because $\{P, Q\}$ and $\{Q, R\}$ are both edges, this sum is strictly more then $||P - Q||_{\vec{y}} + ||Q - R||_{\vec{y}}$. By the triangle inequality, this is at least $||P - R||_{\vec{y}}$. Because $\{P, R\}$ is not an edge, this is at least $d(P, R)$. Having the embedded distance from $P$ to $R$ be more than $d(P, R)$ is illegal. ■

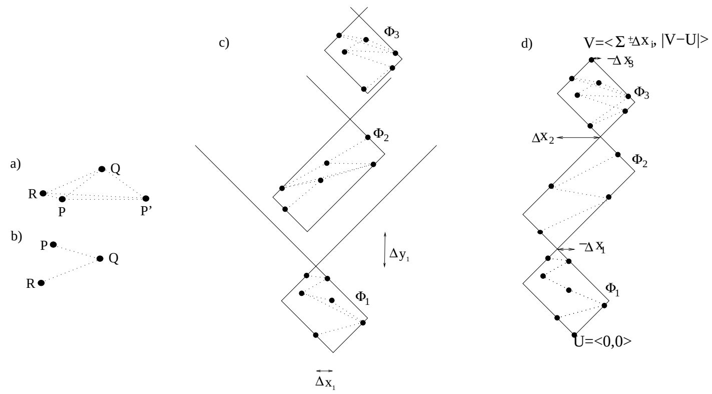


Figure 5: a & b) Fixing points within a component. c) The relative placements of the components $\Phi_1$, $\Phi_2$, and $\Phi_3$. The edges with the components are shown. d) The metric space corresponding to the Subset Sum problem

Each component has now been fixed as a unit. For the remainder of this section we will only consider two dimensions again. The next step is to bound each component $\Phi$ by the smallest rectangle rotated at $45^o$ that contains all the points. See Figure 5.c. Even though they may not be actual points in the metric space, let $bottom(\Phi)$ and $top(\Phi)$ denote the bottom and top corners of this bounding rectangle. For convenience, shift the designated spot within $\Phi$ from that of the first point considered to this bottom corner. This is done by shifting each $X$-location $d_x(\Phi, P)$ within the component. Let us now see how two components can fit together.

**Lemma 4** *Suppose there are are only two dimensions. For every pair of components $\Phi$ and $\Phi'$, one is strictly above the other giving an ordering $\Phi_1, \dots \Phi_J$ of the components along the $Y$ dimension.*

**Proof of Lemma 4:** Consider three points for which $P_y \leq Q_y \leq R_y$. If neither $\{P, Q\}$ nor $\{Q, R\}$ is an edge then $\{P, R\}$ is also not an edge. This is because $R_y - P_y = (R_y - Q_y) + (Q_y - P_y) \geq |R_x - Q_x| + |Q_x - P_x| \geq |R_x - P_x|$. The counter-positive is that if $\{P, R\}$ is an edge, then at least one of $\{P, Q\}$ and $\{Q, R\}$ is. Hence, a component not containing point $Q$ either is strictly above $Q$ or strictly below it. ■

Let us now examine the restrictions on the $X$-coordinates imposed on one component by another.

**Lemma 5** *Suppose that $\Phi'$ is embedded strictly above $\Phi$. Considering only the distances between them, $\Phi$ and $\Phi'$ are free to shift left and right and to flip in the $X$-dimension within the constraint that the rectangle containing $\Phi'$ is embedded above and between the two lines extending upward at $45^o$ from the top two sides of the rectangle containing $\Phi$.*

**Proof of Lemma 5:** Consider the point $Q \in \Phi$ defining the top right edge of the rectangle containing $\Phi$. Consider any point $P \in \Phi'$. In order for the distance $d(P, Q)$ to be manifested in the $Y$ dimension, $P$ must be embedded above and between $Q$'s two upper $45^o$ lines. Hence, $P$ must be above and to the right of the line extending the top right edge of the rectangle. Similarly, it must be above and to the left of the extension of the top left edge. Any other point $Q'$ in $\Phi$ will impose less strict restrictions on the location of $P$. If every point $P \in \Phi'$ is embedded above and between these two $45^o$ lines, then so is the smallest $45^o$ rectangle containing $\Phi'$. ■

Given a line of components, one only need consider the restrictions imposed by consecutive components, because these restrictions are the strongest.

We are now ready to specify the range of possible embeddings for these components. See Figure 4.a. (The case Figure 4.b is similar.) For $i = 1, \ldots, I$, we will embed the component $\Phi_i$. The algorithm started by embedding the point $U$ at location $\varphi(U) = \langle 0, 0 \rangle$. This fixes the location of the bottom most component $\Phi_1$. $\Phi_1$, however, can flip in the $X$-dimension by choosing $flip(\Phi_1) \in \{1, -1\}$.

As a loop invariant, suppose that we have already embedded the components $\Phi_1, \ldots, \Phi_{i-1}$. For $i' < i$, the $X$-coordinate of the bottom corner of bounding rectangle for component $\Phi_{i'}$ is fixed to $bottom(\Phi_{i'})_x$ and $flip(\Phi_{i'}) \in \{1, -1\}$ fixes whether or not $\Phi_{i'}$ is flipped. We continue by letting $\Delta x_{i-1} = d_x(bottom(\Phi_{i-1}), top(\Phi_{i-1}))$, which Lemma 3 gave to be the $X$-distance between the bottom and top corners of the bounding rectangle for $\Phi_{i-1}$. See Figure 5.c. This fixes the $X$-coordinate of the top corner of $\Phi_{i-1}$ to be $top(\Phi_{i-1})_x = bottom(\Phi_{i-1})_x + flip(\Phi_{i-1}) \cdot \Delta x_{i-1}$. We go on to compute $\Delta y_{i-1} = |bottom(\Phi_i)_y - top(\Phi_{i-1})_y|$, which is the known $y$-distance between the top corner of the bounding rectangle for $\Phi_{i-1}$ and the bottom corner of that for $\Phi_i$. Again see Figure 5.c. These two corners cannot deviate by more than this amount in the $X$-dimension. This bounds $bottom(\Phi_i)_x$ to be within the range $[top(\Phi_{i-1})_x - \Delta y_{i-1}, top(\Phi_{i-1})_x + \Delta y_{i-1}]$. We also have freedom to choose $flip(\Phi_i) \in \{1, -1\}$.

This process continues one component at a time until at the top the point $V$ is embedded. We started this process by stating that $V$ is to be embedded at $\varphi(V) = \langle \widehat{x}, \widehat{y} \rangle$, for some unknown value $\widehat{x} \in (x_i, x_{i+1})$. However, at the beginning of this section we relaxed this restriction on $\widehat{x}$.

This completes this embedding. What remains is to check that all the distances are correct. Independent of how the values $bottom(\Phi_i)_x$ and $flip(\Phi_i) \in \{1, -1\}$ are chosen within the above stated constraints, we have fixed the distances $||\varphi(P) - \varphi(Q)||$ between the embedded locations of each pair of points. If we have not already found an inconsistency, then it would now be good to check for every pair of points that this embedded distance is in fact the required distance $d(P, Q)$ given by the metric space. Only after checking this to we accept this embedding.

We started by restricting $\widehat{x}$ to be within one of the ranges $(x_i, x_{i+1})$ (or equal to one $x_i$) and restricting to one of the two embeddings of $W$ (and symmetrically of $\widehat{W}$). After outputting the range of embeddings consistent with these choices, we go on to the next choices.

## 2.3 The Running Time with Different Values of $\widehat{x}$

The above algorithm is challenged because it does not know the value of $V_x = \widehat{x}$. However, the main time that we need this information is to compare it to $v_P - u_P$ for each point $P$. The number of such values $\Delta = \{|v_P - u_P| \mid P \notin \mathcal{S}\}$ is at most $\mathcal{O}(n)$. Hence, the above algorithm needs to be repeated for only $\mathcal{O}(n)$ ranges $(x_i, x_{i+1})$ (or equal to $x_i$). (The value of $\widehat{x}$ being unknown within the range $(x_i, x_{i+1})$ added a few more complications, but we believe these were all handled.)

Consider the total running time. For one interval, placing each point into two and then one region takes only $\mathcal{O}(n)$ time, but to find the components and to check each of the $\binom{n}{2}$ distances requires $\mathcal{O}\left(n^2\right)$ time. This would lead to an $\mathcal{O}\left(n^3\right)$ time algorithm.

This can be improved to a $\mathcal{O}\left(n^2 \log^2 n\right)$ time algorithm as follows. Suppose that we have just completed the algorithm above assuming that $\widehat{x}$ is within one of the intervals and in memory is a data structure describing the situation. When shifting $\widehat{x}$ to the next interval only some of the points will change categories. Across all such shifts a given point $P$ will change categories only twice, namely from $\widehat{x}$ being bigger than $|v_P - u_P|$ to it being equal to it, to it being smaller than it. When a point changes which category it is in, its location might change. Also in the graph used for fixing the last dimension, all the edges adjacent to this node may change. Nothing else will change.

??? Hezinger and King [13] provide a fully dynamic randomized algorithm for maintaining connected components. The total expected time for $p$ edge insertion or deletion updates on an $n$ node graph is only $\mathcal{O}(p \log^2 n)$. In our application, the total number of edge updates is $p = \mathcal{O}(n^2)$ giving that the total time devoted to maintaining the connected components is only $\mathcal{O}(n^2 \log^2 n)$ as required.

The distance between a pair a points need only be rechecked when one of the points changes location. This will occur only $\mathcal{O}(1)$ times per pair for a total of $\mathcal{O}(n^2)$ time. (Actually, when the points $W$ and $\widehat{W}$ move in this way, the complete data structure needs to be changed, however, this occurs only a constant number of times.)

Open problem: Because the edge updates occurs in such an ordered way, is it possible to remove the $\mathcal{O}(\log^2 n)$ factor? Is there a faster way to narrow down the value of $\widehat{x}$ and the placement of $W$ and $\widehat{W}$? Finally, is it really possible to have $\mathcal{O}(n)$ completely different embeddings because of these different initial choices?

## 3 An NP-Completeness Theorem

Theorem 1 proves that a metric space can be embedded into $l^2_\infty$ in time $\mathcal{O}\left(n^2 \log^2 n\right)$. This section will prove that this algorithm is not as flexible to minor changes as we would like.

**Theorem 2** *Embedding a metric space into $l^k_\infty$ is NP-complete given any one of the following.*

1. *The number of dimensions is $k \geq 3$.*
2. *On the $k = 2$ dimensions of the surface of a sphere.*
3. *In $l^2_\infty$ with the added constraint that point $U$ is embedded at $\langle 0, 0\rangle$ and point $V$ embedded at $\langle 0, d(U, V)\rangle$, i.e. $\widehat{x} = 0$.*

We prove these in reverse order.

**Proof of Theorem 2.3:** The reduction is to *Subset Sum*. The input to this problem is a set of positive integers $\{\Delta x_1, \ldots, \Delta x_n\}$. The question is whether there exists a subset $S$ whose sum is equal to the sum of the complement set, i.e., $\widehat{x} = \sum_{i \in S} \Delta x_i - \sum_{i \notin S} \Delta x_i = 0$.

Given an input $\{\Delta x_1, \ldots, \Delta x_n\}$ to Subset Sum, we construct a metric space as follows. Separately for each value $\Delta x_i$, consider a rectangle $\Phi_i$ rotated at $45^o$ such that the difference between the $X$-coordinate of the lower and upper corners is $\Delta x_i$ and the difference between the $Y$-coordinate of the lower and upper corners is more than $\Delta x_i$. See Figure 5.d. Embed in $\Phi_i$ enough points to form a connected component bounded by this rectangle. Place the point $U$ at $\langle 0, 0\rangle$ as required. Place $\Pi_1$ so that its lower corner is on $U$. Stack the rectangles in order on top of each other so that the upper corner for $\Pi_i$ is the lower corner for $\Pi_{i+1}$. (In the notation of Section 2.2, $\Delta y_i = 0$.) Finally, place the point $V$ at the upper corner for $\Pi_n$. The distance between any two points in the metric space are given by this embedding. This embedding, however, will not meet the constraints because $V$ is embedded at $\langle \sum_i \Delta x_i, d(U,V)\rangle$ instead of at $\langle 0, d(U,V)\rangle$.

Recall the embedding algorithm from Theorem 1. A quick check will show that the points $U$ and $V$ are the pair that are farthest apart and that all points are contained within the set $\mathcal{S} = \{P \mid d(U,P) + d(P,V) = d(U,V) = \widehat{y}\}$. This fixes the $Y$-coordinate of each point. Because the set $\{|v_P - u_P| \mid P \notin \mathcal{S}\}$ is empty, the only interval $(\Delta x_i, \Delta x_{i+1})$ within which $\widehat{x}$ needs to be restricted is $(0,\infty)$. Lemma 5 then states that the only degrees of freedom in embedding this metric space is that each component is free flip in the $X$-coordinate, with the corners of consecutive components touching. Hence, there is a one-to-one mapping between the possible embeddings of the metric space and subsets $S \subseteq [1..n]$, where $S$ indicates which rectangles are embedded with their upper corner to the right of their lower corner. Moreover, $V$ is embedded at $\left\langle \sum_{i\in S} \Delta x_i - \sum_{i\notin S} \Delta x_i, d(U,V)\right\rangle$. In conclusion, the metric space can be embedded with $V$ at $\langle 0, d(U,V)\rangle$ if and only if there is a subset of the Subset Sum values $\{\Delta x_1, \ldots, \Delta x_n\}$ for which $\sum_{i\in S} \Delta x_i - \sum_{i\notin S} \Delta x_i = 0$. ■

**Proof of Theorem 2.2:** The only change in the proof required is that the $Y$-dimension cycles around the sphere so that $U$ and $V$ are in fact the same point embedded at $\langle 0,0\rangle = \langle 0, d(U,V)\rangle$. ■

**Proof Sketch of Theorem 2.1:** The proof technique is the same here as well. The only difference is that the circle from $U$ back to $V = U$ travels through two of the three dimensions while the ridged components continue to flip in the $X$ dimension. ■

The complete proof of Theorem 2.1 will be a combination of the proof for Theorem 2.3 and that for Theorem 3. Hence, we will delay it until the end of Section 4.

## 4 3-Dimensional mobius

**Theorem 3** *For every $n \geq 24$, there exists a metric space on $n$ points that cannot be embedded in $l^3_\infty$, however, every proper subspace can be embedded in $l^3_\infty$.*

A similar thing could be proved for dimensions larger than 3. Note that this gives that $c_\infty(3) \geq n-1$.

**Proof of Theorem 3:** We will refer to the metric space in question as the *möbius metric space* because of its relation to a möbius strip. This möbius strip has length traveling around a square within the first two dimensions, $U$ and $V$, and width across the third dimension, $X$. However, along the path around the square, the strip twists connecting the top edge to the bottom and the bottom to the top. This möbius strip cannot be embedded into $l^3_\infty$ since the local information does not allow the strip to flip over. On the other hand, if the möbius strip were cut (by removing points from the metric space), then the strip could be untwisted and embedded into $l^3_\infty$.

We would like the removal of only a single point in the metric space to allow the metric space to be embeddable. Hence, the metric space will have only a single point across the width of the strip whose coordinate in the $X$-dimension is either 1 or -1. Suppose one travels around the square considering the sign of this coordinate. The local distances between consecutive points are able to dictate whether these are the same or the opposite. We will dictate that all consecutive points around the square must have the opposite sign, except at one place around the square, where the consecutive points have same sign. The argument is now a question of parity. It is impossible to have a string from $\{-1,1\}$ of even length in which consecutive entries have opposite signs and the first and the last entries have the same sign. This contradiction ensures that the metric space is not embeddable. On the other hand, if we delete any of the entries from these string requirements, then such a string does exist.

A formal definition of the metric space is as follows. See Figure 6. The $4n$ points are named $\{\langle -n+u, u\rangle \mid 0 \leq u < n\} \cup \{\langle u, n-u\rangle \mid 0 \leq u < n\} \cup \{\langle n-u, -u\rangle \mid 0 \leq u < n\} \cup \{\langle -u, -n+u\rangle \mid 0 \leq u < n\}$. Consecutive points along the square will have distance 2 between them, which is the distance for example between locations $\langle u, v, 1\rangle$ and $\langle u+1, v+1, -1\rangle$, namely $\max(|(u+1)-(u)|, |(v+1)-(v)|, |(-1)-(1)|) = \max(1,1,2) = 2$. The exception is that the distance between the consecutive points $\langle -n+1, -1\rangle$ and $\langle -n, 0\rangle$ is instead 1, which is the distance for example between locations $\langle -n+1, -1, 1\rangle$ and $\langle -n, 0, 1\rangle$, namely $\max(|(-n+1)-(-n)|, |(-1)-(0)|, |(1)-(1)|)\max(1,1,0) = 1$. The distance between non-consecutive points $\langle u, v\rangle$ and $\langle u', v'\rangle$ will be defined to be the distance between the locations $\langle u, v, 1\rangle$ and $\langle u', v', -1\rangle$, which is $\max(|u'-u|, |v'-v|, 2) = \max(|u'-u|, |v'-v|)$. Lemma 7 proves that this metric space cannot be embedded in $l^3_\infty$, while Lemma 6 proves that any proper subspace can be.

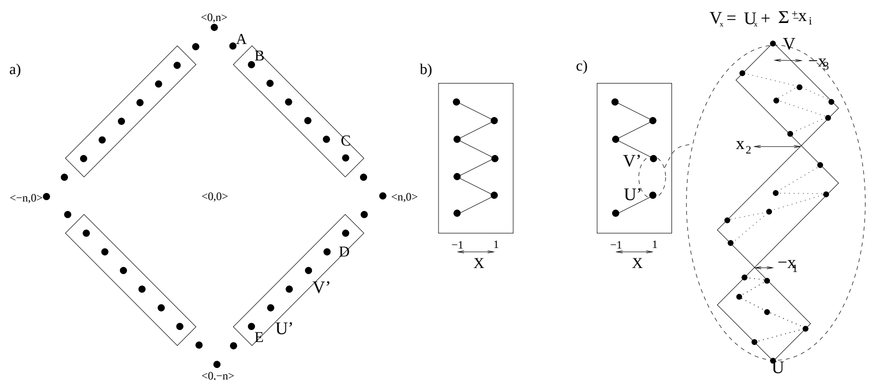


Figure 6: a) In two of the dimensions the points of the möbius metric space form a square. b) Along each edge of the square the points zig zag between $X$-coordinates 1 and $-1$. The edges between the nodes are those used in Lemma 3 for fixing the last dimension. We first prove that the nodes within the long rectangles in both (a) and (b) are fixed to form connected components. Then we show that all the points form a single connected cycle. c) shows how the metric space from Theorem 2.3 is later inserted into this metric space in order to prove Theorem 2.1.

**Lemma 6** *Any proper subspace formed from the möbius metric space by deleting at least one of its underlying points can be embedded in* $l^3_\infty$.

**Proof of Lemma 6:** Let $\langle \hat{u}, \hat{v}\rangle$ be the point that is deleted from the möbius metric space. Each point $\langle u, v\rangle$ is embedded at location $\left\langle u, v, x_{\langle u,v\rangle}\right\rangle$ with $x_{\langle u,v\rangle} \in \{-1, 1\}$. Let $x_{\langle -n,0\rangle} = 1$. Alternate the signs as you go clockwise around the square until you get to the missing point. The pair of entries of distance two across the gap are given opposite signs. After the gap, continue alternating the signs until we are back to $\langle -n+1, -1\rangle$. Note the first and the last points will have the same sign for $X$ as required. This embedding respects all the distances. ■

**Lemma 7** *The möbius metric space defined above cannot be embedded in* $l^3_\infty$.

The following classical *betweenness relation* and lemma will help. We say that the point $B$ *lies between* point $A$ and $C$ if $d(A,B) + d(B,C) = d(A,C)$. For example, all the points in the classification $\mathcal{S}$ of Theorem 1 lie between $U$ and $V$. In $l_2^d$, these points would have to be co-linear. However, this is not the case in $l_\infty^2$.

**Lemma 8** *Consider a metric space and the set of points that lie between a point $A$ to a point $C$. For every embedding of the metric space, modulo translations, renaming of the dimensions, and negations of the dimensions, the coordinates along one of the dimensions are fixed by the distances between the points. More specifically, if the coordinate for $A$ along this dimension is $A_x$, then for the coordinate for any other point $B$ is $B_x = A_x + d(A,B)$.*

Recall that this was done for the points in $\mathcal{S}$ as well.

**Proof of Lemma 8:** Without loss of generality, $C_x = A_x + |C - A|$. If $B_x > A_x + |B - A|$, then $|\varphi(B) - \varphi(A)| = \max(|B_x - A_x|, |B_y - A_A|, |B_z - A_z|) \geq B_x - A_x > |B - A|$, which is a contradiction. Similarly, if $B_x < A_x + |B - A| = C_x - |C - B|$, then $|\varphi(C) - \varphi(B)| = \max(|C_x - B_x|, |C_y - B_y|, |C_z - B_z|) \geq C_x - B_x > |C - B|$. Therefore, $B_x = A_x + |B - A|$. ■

Using Lemma 8, we can say a lot about how the möbius metric space must be embedded.

**Lemma 9** *If the möbius metric space can be embedded into $l_\infty^3$, then without loss of generality for each point except for the eight points adjacent to the corners, $\langle u, v\rangle$ is embedded at location $\left\langle u, v, x_{\langle u,v\rangle} \right\rangle$, for some value $x_{\langle u,v\rangle}$.*

**Proof of Lemma 9:** A quick check will show that all the points except for the four points that are immediately adjacent to the corners $A = \langle -n, 0\rangle$ and $C = \langle n, 0\rangle$ lie between these corners. Therefore, by Lemma 8, without loss of generality for these points, $\langle u, v\rangle$ is embedded at location $\left\langle u, v_{\langle u,v\rangle}, x_{\langle u,v\rangle} \right\rangle$, for some values $v_{\langle u,v\rangle}$ and $x_{\langle u,v\rangle}$. Going the other direction, the distance between the corners $A' = \langle 0, -n\rangle$ and $C' = \langle 0, n\rangle$ cant be manifested in the same dimension. Hence, applying Lemma 8 again gives us that for each point except for the eight points adjacent to the corners, $\langle u, v\rangle$ is embedded at location $\left\langle u, v, x_{\langle u,v\rangle} \right\rangle$, for some value $x_{\langle u,v\rangle}$. ■

**Lemma 10** *Points at even distance along the same side of the square, excluding the corners and the two points that are adjacent to them, are embedded at locations with the same $X$-coordinate.*

**Proof of Lemma 10:** The $U$ and $V$ coordinates of the points in question have been fixed. For these points, this leaves only one dimension $X$ undetermined. Lemma 3 describes how to partition these points into connected components, each of whose embedding is fixed. In this graph, consecutive points, which have distance 2 between them but only distance 1 in the $\langle U, V\rangle$ dimensions, have edges between them. It follows that each of the four sides of the square, excluding the corners and the two points that are adjacent to them form components. See Figure 6:a and b. The lemma follows. ■

Lemma 9 failed to consider the eight points adjacent to the corners. We are now ready to consider these.

**Lemma 11** *For the eight points adjacent to the corners, $\langle u, v\rangle$ is embedded at location $\left\langle u, v, x_{\langle u,v\rangle} \right\rangle$, for some value $x_{\langle u,v\rangle}$.*

**Proof of Lemma 11:** By symmetry of the argument, consider the point denoted $A = \langle 1, n-1\rangle$. From the proof of Lemma 9, we know that it is embedded at location $\langle 1, A_v, A_x\rangle$ for some values $A_v$ and $A_x$. Our goal is to prove that $A_v = n-1$. See Figure 6:a. If $n$ is even, consider the sequence of points $A$, $B = \langle 2, n-2\rangle$, $C = \langle n-2, 2\rangle$, $D = \langle n-2, -2\rangle$ and $E = \langle 2, -n+2\rangle$. If $n$ is odd, instead let $C = \langle n-3, 3\rangle$ and $D = \langle n-3, -3\rangle$. $|A_x - B_x| \leq d(A,B) = 2$ and $|C_x - D_x| \leq d(C,D) \leq 6$ by the given distances. $|B_x - C_x| = |D_x - E_x| = 0$, by Lemma 10. Hence, $|A_x - E_x| \leq |A_x - B_x| + |B_x - C_x| + |C_x - D_x| + |D_x - E_x| \leq 2+0+6+0 = 8$. We also have that $|A_u - E_u| = |1-2| = 1$. It follows that the given distance $d(A,E) = 2n-3$ is manifested in the second dimension, i.e., $|A_v - E_v| = |A_v - (-n+2)| = 2n-3$. Clearly, $A_v$ is not smaller then $-n$, concluding that $A_v = n-1$. ■

Now that the $U$ and $V$ coordinates have been fixed for all of the points in the metric space, we are ready to apply Lemma 3 again. Because consecutive points (except for the one pair that has distance 1 between them) have an edge between them in the component graph, the entire square becomes one component. Hence, without loss of generality, all $X$-coordinates are $x_{\langle u,v\rangle} \in \{-1, 1\}$ and consecutive points with distance 2 must have opposite signs, while the one pair of consecutive points with distances 0 must have the same sign. As said initially, this is impossible. This concludes the proof of Theorem 3. ■

We are now ready to complete the remaining proof from Section 3 that embedding a metric space into $l_\infty^3$ is NP-complete.

**Proof of Theorem 2.1:** Given an instance $\{\Delta x_1, \ldots, \Delta x_n\}$ to Subset Sum, we construct a metric space as follows. See Figure 6:c. Start by building the möbius metric space from Theorem 3. To create the twist in this möbius strip, the distance between the consecutive points $\langle -n+1, -1\rangle$ and $\langle -n, 0\rangle$ was defined to be 1 instead of the usual 2. We remove this twist by changing the distance to be 2. Then let $U'$ denote the point $\langle 3, -n+3\rangle$, let $V'$ denote $\langle 5, -n+5\rangle$, and remove the point $\langle 4, -n+4\rangle$ between them. Just as was done for Theorem 3, we can prove that without loss of generality each point $\langle u, v\rangle$ is embedded at location $\left\langle u, v, x_{\langle u,v\rangle}\right\rangle$, where $x_{\langle u,v\rangle} \in \{-1, 1\}$ and consecutive points have opposite signs. The only difference is that because the point $\langle 4, -n+4\rangle$ has been removed, the points $U'$ and $V'$ are in the same connected component only by following the path the long way around the square.

The statement of Theorem 2.3 requires the added constraint that point $U$ is embedded at $\langle 0, 0\rangle$ and point $V$ embedded at $\langle 0, d(U,V)\rangle$. Instead, we have that point $U'$ is embedded at $\langle 3, -n+3, 1\rangle$ and point $V$ embedded at $\langle 5, -n+5, 1)\rangle$.

Given the instance $\{\Delta x_1, \ldots, \Delta x_n\}$ to Subset Sum, construct the metric space as done in the proof of theorem 2.3. Scale all the distances down so that $d(U,V) = 2$. To combine these two metric spaces it is only necessary to give the distances between each point in the first and each in the second. Imagine rotating the second metric space by $45^o$ and inserting it in the first equating point $U$ with $U'$ and $V$ with $V'$. This requires having the $Y$ axis of the first metric space be rotated to go along the $45^o$ line between points $U'$ and $V'$. Once this is done define the distances between the new pairs of points to be their distances determined by $u$ and $v$ axis.

Just as done in the proof of Lemmas 9-11, we can prove that if this combined metric space can be embedded into $l_\infty^3$, then without loss of generality the $u$ and $v$ coordinates of each point has been fixed leaving only one dimension $X$ undetermined. Lemma 3 describes how to partition these points into connected components, each of whose embedding is fixed. As is true in Theorem 3, there will be one component going the long way around the square from $U'$ to $V'$. As is true in Theorem 2.3 there will be one component for each $\Delta x_i$ value. Though Lemma 5 was actually proved for only two dimensions, with added conceptual difficulty it could be extended to three dimensions. However, this is not really necessary because our string of components do lie in two dimensions, one being the line from $U'$ to $V'$ and one being the $X$-dimension. Hence, we will freely use Lemma 5 to prove that the only degrees of freedom in embedding this metric space is that each component is free flip in the $X$-coordinate, with the corners of consecutive components touching. As was true with Theorem 2.3, there is a one-to-one mapping between the possible embeddings of the metric space and subsets $S \subseteq [1..n]$, where $S$ indicates which rectangles are embedded so that the $X$-coordinate from one corner to the other is increasing or decreasing by $\Delta x_i$. Because $U'_x = V'_x = 1$, the metric space can be embedded if and only if there is a subset of the Subset Sum values $\{\Delta x_1, \ldots, \Delta x_n\}$ for which $\sum_{i \in S} \Delta x_i - \sum_{i \notin S} \Delta x_i = 0$.

If the number of dimensions $k$ is more three, then the same proof holds after adding a point far in the positive direction and one far in the negative direction for each of the extra dimensions. The distances to these points can be used to fix the coordinates in all but three of the dimensions to zero. ■

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## References


[1] P. Assouad and M. Deza. “Espaces metriques plongeables dans un hypercube: aspectes combinatoires”, *Annals of Discrete Mathematics* 8 (1980), 197–210.

[2] P. Assouad and M. Deza. “Metric Subspaces of $L^1$”, *Publications Mathematiques D’Orsay,* Universite d Paris-Sud, Orsay (1980).

[3] D. Avis and M. Deza, The Cut Cone, $L_1$Embedability, complexity and multicommodity flows, *Networks*, Vol. 21, pp. 595–617 (1991).

[4] M. Badoiu, “Approximation algorithm for embedding metrics into a two-dimensional space”, *SODA’03*

[5] H.J. Bandelt and V. Chepoi, “Embedding metric spaces in the rectilinear plane: a six–point criterion”, *Discrete Computational Geometry* 15 (1996), 107-117

[6] H.J. Bandelt and V. Chepoi, “Embedding into the rectilinear grid”, *Networks* 32(1998), 127-132

[7] H.J. Bandelt and V. Chepoi, M. Laurent, “Embedding into rectilinear spaces”, *Discrete Computational Geometry* 19(1998), 595-604)

[8] L. Blumenthal, Distance Geometries, *University of Missouri Studies*, vol. 13 no. 2 (1938)

[9] L. Blumenthal, Theory and Applications of Distance Geometry, *Oxford University Press,* (1953)

[10] G.E. Christopher and M.A. Trick, “Faster decomposition of totally decomposable metrics with applications”, 1996, Carnegie Mellon University.

[11] M. Deza and M. Laurent, “Geometry of Cuts and Metrics is a good general reference on metrics and isometric embeddings”, book.

[12] Jeff Edmonds, “Embedding into $l^2_\infty$ is Easy, Embedding into $l^3_\infty$ is NP-Complete,” Earlier version of this same paper. *SODA 2007.*

[13] M. Henzinger and Valerie King, ”Randomized Dynamic Graph Algorithms with Polylogarithmic Time per Operation” *(STOC’95). Journal of the ACM*, Vol. 46 No. 4 (1999) pp. 502–516.

[14] S. Malitz and J. Malitz, “A bounded compactness theorem for $L^1$-embeddings of metric spaces in the plane”, In *Discrete Comput. Geom.*, 8 (1992) pp. 373–385.

[15] J. Matousek “Open problems on embeddings of finite metric spaces”, personal web page.

[16] K. Menger, Untersuchungen uber allgemeine Metrik, *Mathematishe Annalen* 100 (1928), pp. 75–163.

[17] James Schmerl, private communication with Seth Malitz and Jerome Malitz (1990).