Momentum, Kinetic Energy, and Temperature

Start with Newton's three laws: 1 - things continue to traveling in a straight line, 2 - pushing things makes them accelerate $\vec{f} = m\vec{a}$, and 3 - colliding particles apply forces to one another that are equal in magnitude and opposite in direction. Add a desire for conservation laws and that randomness uniformly distributes the particles of a gas. From these alone our goal is to develop and relate momentum $\sum_i m_i\vec{v_i}$, potential energy $E_p = fd = mad$, kinetic energy $\sum_i m_i{v_i}^2$, thermal energy, temperature, and gas pressure. I have not really read up on it, because I like doing things myself. I heard that the math was hard, but unless I am doing it wrong, it is quite easy. I would love it if you loved it too.

My problem for years was that force relates to the rate of momentum transfer which depends on $\text{Avg}_i v_i$, while kinetic energy depends on $\text{Avg}_i {v_i}^2$ and that these two values and can be vastly different values depending on whether all the particles have the same velocity or one has N times this velocity and the other $N-1$ have $v_i = 0$. I was happy when Janos told me that we should assume that the velocities of particles are in the Boltzmann folded normal distribution which gives $(\text{Avg}_i|v_i|)^2 = \frac{2}{\pi}\text{Avg}_i {v_i}^2$. I was quite pleased, however, that this was not needed. The force exerted by a gas per unit area is the kinetic energy per the unit volume which is the temperature of the gas. This requires that the particles are distributed uniformly but assumes nothing about the distribution on the velocities.

The average momentum of the micro particles in the gas is defined as macro value $\text{Avg}_i m_i\vec{v_i}$ because this is a value that is conserved as the particles collide. If the macro gas is not flowing then this macro value is zero because all the vectors being in different directions cancel out. We shift this to average speed $\text{Avg}_i v_i$ by taking the component of each momentum in the direction of the container. This becomes $(\text{Avg}_i v_i)^2$ when we include both the momentum of the particle and the rate at which they collide against the container. This becomes $\text{Avg}_i {v_i}^2$ because these two events are correlated. The particles that have a bigger speed v_i both contribute more momentum and do so faster. The same sum of squares is also the definition of kinetic energy because this is the definition that allows for the conservation of potential vs kinetic energy. This kinetic energy causes the gas to put pressure on its container making the mercury expand defining as such temperature measured by a thermometer. This gives rise to the conservation of kinetic vs thermal energy. Cool.

Defining Momentum, Potential, and Kinetic Energy

Particle Velocities: Let $\vec{v_i}$ denote the velocity of the i^{th} particle in the gas as a direction & speed vector. Let $v_i = |\vec{v_i}|$ denote its $meters/sec$ speed.

$\boldsymbol{\vec{f} = m\vec{a}}$**:** Experimentation revealed that if you have twice as many people pushing then the car accelerates twice as fast and that if the car weighs twice much then it accelerates half as fast. The constant 1 is set by defining the unit newton (N) of force represents the force required to accelerate a mass of one (kg) by one (m/s^2).

Momentum: On a micro level, we may not know the direction or speed of each particle in a gas, but what macro number will stay constant as the particles collide? When two collide, Newton stipulates that the force $\vec{f}$ of the first exerted on the second is equal to the force $-\vec{f}$ of the second exerted on the first but in the opposite direction. By $\vec{f} = m\vec{a}$, this will accelerate them at a rate of $\vec{a}_i = \frac{1}{m_i}\vec{f}$. If the duration of this force is δt, then the change in velocities will be $\delta\vec{v}_i = \vec{a}_i \cdot \delta t = \frac{1}{m_i}\vec{f}\delta t$. To make this the same for the two particles lets multiply this by their individual masses, namely $\delta\vec{p}_i = m_i\delta\vec{v}_i = \vec{f}\delta t$. Because this vector is in the opposite direction, these two changes sum to zero. This motivates defining a particles momentum to be $\vec{p_i} = m_i\vec{v_i}$. We just proved that total momentum of the system $\sum_i \vec{p_i} = \sum_i m_i\vec{v_i}$ remains constant as the particles elastically collide,

A Vector: Momentum $m_i\vec{v_i}$ has both a magnitude and a direction. If the gas as a macro system is not moving, then these momentum vectors which point in all directions will cancel out given a total momentum of $\sum_i m\vec{v_i} = 0$. This is not helpful to us in our task.

Energy: Newton would have had the conceptual understanding that energy is what is needed to get work done. Potential energy $E_p = fd = mad$ arises when a particle is positioned a distance d up a force field (eg gravity) and hence has the potential to accelerate through this distance creating velocity. Kinetic energy $E_k = \frac{1}{2}mv^2$ arises when a particle is moving at velocity v and hence can do "work" by smashing into something. These clearly depend linearly on the mass m, because if you glue two particles together the energy doubles. The first clearly depend linearly on the distance d, because if you move it this distance once and then a second time, it clearly takes twice the energy. The constant 1 in the first is set by defining the units of energy as such. But why are they linear in a and v^2? Could we just as well define $E_p = ma^3d$ and $E_k = 7mv^5$?

Conservation of Energy: I have generally seen that $E_p = mfd = ad$ and $E_k = \frac{1}{2}mv^2$ are given as assumptions from which one proves conservation of energy. I prefer to turn it around. We want conservation of energy and these are the only two equation that give it it you.

By this we mean the following. Let d_i, v_i, m, and a be the initial distance from the ground, vertical velocity, mass, and downward acceleration. After time t, calculate the new velocity to be $v(t) = v_i + at$ and the distance traveled to be $\Delta d(t) = \int_t v(t)\delta t = \int_t (v_i + at)\ \delta t = v_i t + \frac{1}{2}at^2$. Calculate the change in potential energy $\Delta E_p = ma\Delta d(t) = ma(v_i t + \frac{1}{2}at^2)$ and the change in kinetic energy $\Delta E_k = E_k(t) - E_k(0) = \frac{1}{2}mv(t)^2 - \frac{1}{2}mv_i{}^2 = \frac{1}{2}m(v_i + at)^2 - \frac{1}{2}mv_i{}^2 = \frac{1}{2}m(2v_i at + a^2t^2) = ma(v_i t + \frac{1}{2}at^2) = \Delta E_p$. Note that these definition make $\Delta E_p = \Delta E_k$ conserving energy. Playing a little more, one can see that other definitions of energy would not do this.

Angular Momentum: Kepler said that the angle swept out by each planet each second remains a constant. This is the conservation of angular momentum. It means that when a spinning person brings in their hands, they spin faster. Newton has a fantastic proof involving the area of triangles. I would love to show it to you.

Elliptical Orbit: I also made power point slides to give a geometric proof that the orbit of plants is elliptical with the sun at the focal point.

Force per Unit Area Exerted by a Gas onto its Container is Equal to its Average Particle Kinetic Energy per Unit Volume

ChatGPT gave me the law

$$P = \frac{2}{3}\frac{N}{V}E_k$$

but says that the proof is too hard to present. Here pressure P is the force per unit area exerted by the gas, N is the number of gas molecules in the container, V its volume and E_k is the average kinetic energy of one particle of the gas. If we assume that the particles are always uniformly distributed, then there are $\frac{N}{V}$ particles per unit volume and their expected total kinetic energy per unit volume is $\frac{N}{V}E_k$. The only assumption about the particles is that their location is always uniformly distributed through the space. As I had first thought, we do not need to assume anything about the distribution of the particle velocities. (Our calculations are off by the $\frac{2}{3}$ factor.)

Pressure: Pressure P is the force per unit area exerted by the gas on its container from particles hitting it. If you punched a hole anywhere in the container, the gas would come streaming out with a force F proportional to the area A of the hole, i.e., $F = PA$. One Pascal (Pa) of force is defined as one newton of pressure per square meter (N/m^2), while a newton (N) is the force required to accelerate a mass of one kilogram (kg) by one meter per second squared (m/s^2).

Rate of Transfer of Momentum Gives Force: We know $\vec{f} = m\vec{a}$ because pushing things makes them accelerate. We know acceleration is the rate of change of velocity, giving $\vec{f} = m\frac{\delta \vec{v}}{\delta t} = \frac{\delta m\vec{v}}{\delta t}$. Having defined $m\vec{v}$ to be momentum, this gives that the rate that the momentum of an object changes is equal to the force at which you push it. It is fun that this same "per time" unit can arise in a seemly different way. Fire a stream of particles at the wall of a container at some rate, i.e., the number of particles per second. The momentum mv of each particle gets transferred to the container wall. This leads to a rate of transfer of momentum onto the container. It turns out that this is also equal to the force exerted onto a contain by the stream of particles.

Area Region A: Let A denote both an area on the (bottom) surface of the container and its area. Our goal is to argue that the force exerted by the gas on it is equal to A times the average particle kinetic energy per unit volume.

Area A times length L equals Volume: Let AL denote the 3-dimensional space on top of area A with height L and volume AL. If we assume that the particles are always uniformly distributed, then there are $AL\frac{N}{V}$ particles within this volume AL. If such a particle is located at the top and travels straight down, then it must travel a distance L in order to collide with the container at area A.

Particle Collisions: As a particle travels, it collides with another particle every few nano meters (billionths of a meter) making its path very hard to predicted. To avoid this, we will make the volume AL infinitesimally small so that the expected number of particles in it is much less than one. This makes $AL\frac{N}{V}$ no longer be the number of particles within this volume AL but instead the probability that there is a single particle in this volume. Because there is not a second particle anywhere near the first during the time and space considered, we do not have to consider particle collisions at all.

Particle i: Assuming the volume AL contains the i^{th} particle in the gas, let $\vec{v_i}$ denote its velocity as a direction & speed vector and $v_i = |\vec{v_i}|$ denote its $meters/sec$ speed. Let $L_i \in [0, L]$ denote its distance to the container at the beginning of the time period considered. With probability $\frac{1}{2}$, it is headed towards the container wall and not away from it. Assuming the former, let θ_i denote the angle between $\vec{v_i}$ and the vertical so that the distance the particle needs to travel along the hypotenuse to reach the container is $\frac{L_i}{cos(\theta_i)}$. Lets assume that A is big compared to L, so that the probability of the particle hitting the container outside of area A is insignificant.

Rate: Traveling at speed v_i, the time until the collision with the container is $\frac{L_i}{v_i cos(\theta_i)}$ and hence the "rate" of collisions per second is $\frac{v_i cos(\theta_i)}{L_i}$.

Momentum: The momentum of the particle is $m_i\vec{v_i}$, but only the component $m_i v_i cos(\theta_i)$ perpendicular to the container is transferred to the container.

Concluding $P = \frac{2}{3}\frac{N}{V}E_k$: The force exerted onto a contain by the gas is equal to rate of transfer of momentum onto the container. We obtain this rate by putting these pieces together: the probability $AL\frac{N}{V}$ of there being a point in this volume; the probability $\frac{1}{2}$ that the particle is going down; the rate $\frac{v_i cos(\theta_i)}{L_i}$ of this particle colliding against the container; and the transferred momentum $m_i v_i cos(\theta_i)$, gives a rate of momentum transfer of $AL\frac{N}{V} \times \frac{1}{2} \times \frac{v_i cos(\theta_i)}{L_i} \times m_i v_i cos(\theta_i)$. Plugging in $Exp(L_i) = \frac{1}{2}L$ and $Exp(cos^2(\theta_i)) = \frac{1}{2}$ gives $A\frac{N}{V} \times \frac{1}{2}m_i {v_i}^2$. If you average this over all the gas particles, one gets A times the average particle kinetic energy per unit volume as needed for the right hand side of the theorem. As said, these calculations do not give the constant $\frac{2}{3}$ given by ChatGPT.

Boltzmann Folded Normal Distribution

I was happy when Janos told me that we should assume that the velocities of particles are in the Boltzmann folded normal distribution.

Tending to Gaussian/Normal: Here is a fun fact. Take any very long sequence of independent random experiments. Each can have what ever distribution on the real numbers that you want. Let Sum denote the random variable obtained by adding of the result of these. As the sequence gets longer, this sum distribution tends to be normally distributed.

Random Collisions: I image each collision of two particles is an independent experiment that distributes the velocities from one particle to another. Perhaps the final velocity v_i is the "sum" of these experiments.

Folding the Distribution: The velocity $\vec{v_i}$ is in a random direction. If the macro system of gas is at rest, then the average momentum is zero. If all the weights are the same m, then the average velocity is zero, Avg$\vec{v_i} = 0$. If this direction was projected onto a line, then v_i would be a random positive and negative real number. The claim is that this value is normally distributed with mean $\mu == 0$. To get speed, we take the absolute value $|v_i|$. Wiki calls this the folded normal distribution.

Boltzmann: Wikipedia gives that the probability that the i^{th} particle has a given velocity v is proportional to $Prob(v_i = v) \;\propto\; e^{-\frac{E_k(v)}{k_B T}} = e^{-\frac{\frac{1}{2} m_i v^2}{k_B T}}$ where $k_B = 1.380649 \times 10^{-23}$ is the Boltzmann constant and T is the temperature in Kelvin.

Folded Normal: To me this distribution is the folded normal distribution, namely, let v_i be a normally distributed random variable $Prob(v_i = v) \;\propto\; e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, with mean $\mu = 0$ and variance $\sigma^2 = \frac{k_B T}{m_i}$.

Variance: The variance is defined $\text{Exp}(v_i{}^2)$. Multiplying by $\frac{1}{2} m_i$ gives that $\text{Exp}(E_i) = \text{Exp}(\frac{1}{2} m_i v_i) = \frac{1}{2} m_i \sigma^2 = \frac{1}{2} k_B T$.

Kinetic Energy $\propto$ Temperature: This tells that that the expected kinetic energy of a particle is proportional to its temperature.

Avg$|v_i|$: We "expect" a normal variable to be 1, 2, or 3 standard deviations from the mean. Namely $\text{Avg}|v_i|$ is going to be a few standard deviations, The standard deviation σ squared is the variance, Wikipedia confirms this with $(\text{Avg}_i |v_i|)^2 = \frac{2}{\pi} \text{Avg}_i v_i{}^2$. Note that both sides need to have a square in them to get the units right.