Reuleaux = Equal Distant Rolling thing

<https://en.wikipedia.org/wiki/Reuleaux_triangle>

At lunch Saturday, we considered 2 dimensional objects with the following amazing properties.
Roll this shape on a table with a book on top so that the book is always perpendicular to the table.
The distance between the table and the book remains constant as we roll along.
I would have tried to prove that the circle is the only example of this.
But Alex gifted Jack a different shape.

Here is a generalization of the Equal Distant Rolling thing.
Martin: I wonder if your ideal of things warping continually from a circle to the triangle example
is demonstrated in this construction.


Follow these steps:

1. Get a stick of any fix length to be our "diameter".
2. Permanently attach a pencil at both ends of the stick so that whenever the stick slides on the paper it draws a curve.
3. Place the stick on paper. Draw this line segment traced by the stick. Label its ends  a1 and b1.
4. Loop i=1,2,3,...,n
5. Loop Invariant: The stick is on a line segment drawn on the paper with ends a\_i and b\_i
6. Choose any point c\_i between a\_i and b\_i. Mark this point both on the paper and on the stick.
7. Rotate the stick clockwise pivoting at this point c\_i through any angle of your choice.
8. When you stop rotating, draw this line segment traced by the stick. Label its ends  a\_(i+1) and b\_(i+1).
9. Note that we traced a curve from a\_i to a\_(i+1) and from b\_i to b\_(i+1).
10. Go to step 4 and repeat.
11. Hard part. Make sure the shape stays convex.
12. Hard part. The final points a\_n and b\_n must match up with the first points b\_1 and a\_1 but switched.

Note a circle always puts c\_i in the middle of a\_i and b\_i.
The triangle example has c\_i alternate between being a\_i and b\_i.

Claim: This is a generalization of the Equal Distant Rolling thing

Proof:

1. Roll this shape on a table with a book on top so that the book is always perpendicular to the table.
2. Stop rolling arbitrarily.
3. Call the point that is currently touching the table point a.
It is some point along the surface curve of the shape.
4. Place the stick where it was when this point a was drawn.
5. Let b be the other end of the stick when at this location.
6. Let c be the point c\_i that the stick was rotating on at this time.
7. Note that the curve containing a is an arch of some fixed radius centered at c.
8. Hence, the stick currently is perpendicular to this curve at a.
9. Hence, the stick is perpendicular to both the table and the book.
10. Similarly, the stick is perpendicular to the curve at b.
11. Because the shape is convex, the book must be touching point b.
12. Hence, the distance between the table and the book is equal to the length of the stick.
13. Hence, the distance between the table and the book remains constant as we roll along.

QED?

1. There remains the transitions as the center c\_i changes to c\_(i+1).
If both c\_i and c\_(i+1) are on the interior of the stick,
then the curve, being perpendicular from both directions,
must be smooth. It seems ok.
2. Now suppose c\_i is equal to a\_i.
Then the curve is not smooth at a\_i.
Because the stick keeps rotating clockwise, I think that the curve is convex.
But this is the triangle example and I dont see anything that goes wrong.
3. Certainly, distance between the table and the book is the same rolling up to this point and away from it.

Open question: If we can change the center from c\_i to c\_(i+1) any finite number of times,
what happens if it changes continuously?

I figured out how to match up the ends by changing any two of the steps.  And that it is always convex.

You can give me any set of parameters.

- the length of the stick

- the initial location of the stick

- these number n of steps

- for each step i

   - the angle through which the sick rotates

   - the location ci on the stick at which the sick pivots

And you specify any two of the steps.

Then i can change only those two steps to make it all work.

The only caveat is the danger that i not make the angles of rotation go negative or the pivot locations not shift on the stick past its ends. But this should be avoidable with more thought

The stick needs to rotate continually for 180 degrees.  Hence these step angles need to be positive and add up to 180.

We were focused on whether the ends of the stick in its last location aligne with that of the its first location.  Instead let's denote the x-y coordinates of the location of the center of the stick in its ith location to be di.  Because we already ensured that the sticks initial and final rotations align, it is sufficient to make the first and last centers be the same ie d0 equals dn.

To this end, I start by conducting the shape according to your parameters and comparing the d0 and dn that arise. Let v denote the vector from your d0 to your dn.

My job now is to change ci rotation locations for the two specified steps so that your dn shifts through this vector v so that it aligns with your d0.

Im on my phone at the cottage so i can't draw you a picture. But let me talk you through it.

Let step i be one of the two whose ci we are allowed to change.  Reset the pivot location ci to be in the center of the stick.  This has the effect that the center of the stick does not move when the stick rotates. Ie di = di+1.

Draw the stick at its ith rotation and at its i+1st location intersecting in their centers di. Draw the line through di that is perpendicular to the average of the two stick locations.  Denote this line Ci. To check that you have done it right, any line parallel to Ci forms an equilateral triangle with the two stick locations.

Recall that our task is to choose a new pivot location ci. As practice, try out a few values for ci. The claim is that the next stick center di+1 will always lie on this line Ci. In fact we can move di+1 to be any where we choose along line Ci with the only caviat being the danger that we want to move the pivot ci beyond the ends of the stick. Let vi denote the unit length vector indicating the direction of line Ci.

Similarly let vj be the vector for the step j that we are also allowed to change. Note that in different steps the sticks will have different rotations and hence vi and vj do not point in the same direction. Ie linearly independent.

Now note that changing ci and cj has the effect of changing the location of the final stick center location dn in the same way.  In fact changing ci moves dn arbitrarily far in the direction indicated by vi. Similarly changing cj moves dn in the direction of vj. Recall that our goal is to move dn by the vector v. Using linear algebra we can produce the effect of moving dn by vector v by moving some distance in the vi direction and some distance in the vj direction.  This lines up our initial and final stick centers ie c0=Cn. This lines up the stick's end points.  And our shape is done.

The next step it to make sure that the shape is convex. This is easy.  We saw that the curve follow the end of the stick in the direction perpendicular to the stick's rotation. The stick's rotation always increases clockwise.  Hence so does the direction of these curve.  Hence the shape is convex.

We can finally argue that all two dimensional rolling things can be formed this way. Consider a rolling thing rolled to some point.  We claim the rolling thing is touching the table and the book at one point each one directly above the other at the distance of the stick.  Otherwise this non perpendicular distance between these points on the rolling thing would be too big.

Big love Jeff