Lecture #19, Nov 20, 2020

0.0.1 Example. Consider the wff

$$x = y \to (\forall x)x = y \tag{1}$$

Here are a few interpretations:

1. $D = \{3\}$, $x^{\mathfrak{D}} = 3$, $y^{\mathfrak{D}} = 3$.

   Since D contains one element only the above "choice" was made for us, being unique.

   Thus (1) translates as

   $$3 = 3 \to (\forall x \in D)x = 3 \tag{2}$$

   Incidentally, (2) is TRUE.

2. This time I take

   $D = \{3, 5\}$, and again $x^{\mathfrak{D}} = 3$ and $y^{\mathfrak{D}} = 3$.

   Thus (1) translates as:

   $$3 = 3 \to (\forall x \in D)x = 3 \tag{3}$$

   This time (3) is FALSE since "$3 = 3$" is TRUE as before, BUT

   $$\text{"}(\forall x \in D)x = 3\text{" is FALSE.}$$

$\square$

0.0.2 Example. Let's interpret the following a few different ways:

$$(\forall x)(x \in y \equiv x \in z) \to y = z \tag{1}$$

1. First this is true if we really are talking about sets as "$\in$" compels us to think, being THE predicate of set theory that says "*is a member of*".

   Incidentally, (1) if interpreted in Set Theory, says that any two sets y and z are equal if they happen to have the same elements (x is in y iff x is in z). *Hence is true, as I noted.*

2. Let us now interpret in number theory (of $\mathbb{N}$).

   Take $D = \mathbb{N}$ and $\in^{\mathfrak{D}} = <$, where "$<$" is the relation "*less than*" on $\mathbb{N}$.

Wait a minute! Can I do that?! Can I interpret "$\in$" as something OTHER than "*is a member of*"?

Of course you can!

Only "$=, (,), \neg, \vee, \wedge, \rightarrow, \equiv$" translate as themselves!

EVERYTHING ELSE is fair game to translate as you please!

So (1) translates as:

$$(\forall x \in \mathbb{N})(x < y^{\mathfrak{D}} \equiv x < z^{\mathfrak{D}}) \rightarrow y^{\mathfrak{D}} = z^{\mathfrak{D}}$$

which is TRUE no matter how we choose $y^{\mathfrak{D}}$ and $z^{\mathfrak{D}}$.

3. Next, let $D = \mathbb{N}$ and $\in^{\mathfrak{D}} = \,|$, where "$|$" indicates the relation "*divides*" (with remainder zero).

   E.g., $2 \,|\, 3$ and $2 \,|\, 1$ are FALSE but $2 \,|\, 4$ and $2 \,|\, 0$ are TRUE. Then (1) translates as:

   $$(\forall x \in \mathbb{N})(x \,|\, y^{\mathfrak{D}} \equiv x \,|\, z^{\mathfrak{D}}) \to y^{\mathfrak{D}} = z^{\mathfrak{D}}$$

   which is also TRUE for all choices of $y^{\mathfrak{D}}, z^{\mathfrak{D}}$.

   It says: "*Two natural numbers, $y^{\mathfrak{D}}$ and $z^{\mathfrak{D}}$, are EQUAL <u>if</u> they have exactly the same divisors*".

4. But consider something slightly different now: Take $D = \mathbb{Z}$ —the set of all integers— and $\in^{\mathfrak{D}} = \,|$. Take also $y^{\mathfrak{D}} = 2$ and $z^{\mathfrak{D}} = -2$.

   Then (1) translates as

   $$(\forall x \in \mathbb{Z})(x \,|\, 2 \equiv x \,|\, -2) \to 2 = -2$$

   This is FALSE, for 2 and -2 have the same divisors, but $2 \neq -2$.

   So (1) is <u>NOT TRUE IN ALL INTERPRETATIONS</u>.

$\square$

0.1.1 Definition. (Universally Valid wff)

Suppose that $A^{\mathfrak{D}} = \mathbf{t}$ for some A and $\mathfrak{D}$.

We say that A *is true in the interpretation* $\mathfrak{D}$ or that $\mathfrak{D}$ *is a model of* A.

We write this thus:

$$\models_{\mathfrak{D}} A \tag{1}$$

A 1st-order wff, A, is *universally valid* —or just "valid"— iff *EVERY interpretation of the wff is a model of it*, that is, we have that (1) *holds for every interpretation* $\mathfrak{D}$ of the language of A.

In symbols,

$$A \text{ is valid iff, for all } \mathfrak{D}, \text{ we have } \models_{\mathfrak{D}} A \tag{2}$$

(2) has the short expression (3) below:

$$\models A \tag{3}$$

A formula A that satisfies (3) is sometimes also called Logically or Absolutely valid. □

0.1.2 Remark. NOTE the absence of the subscript "*taut*" in the notation (3) above.

The symbols $\models$ and $\models_{taut}$ are NOT the same!

For example, $\mathbf{x} = \mathbf{x}$ translates as

$$\mathbf{x}^{\mathfrak{D}} = \mathbf{x}^{\mathfrak{D}} \tag{4}$$

in EVERY interpretation $\mathfrak{D}$, and is thus true in every interpretation, since it is a self-evident philosophical truth that every object is equal to itself!

Thus, we have $\models \mathbf{x} = \mathbf{x}$.

On the other hand, $\models_{\mathrm{taut}} \mathbf{x} = \mathbf{x}$ is NOT a TRUE meta statement.

$\mathbf{x} = \mathbf{x}$ is NOT a tautology! It is a prime formula (WHY?) hence a Boolean variable!

NO Boolean variable is a tautology as I can assign to it the VALUE FALSE. □

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Valid Axioms 1. Ax1. Every axiom here is a tautology A. Thus $\models_{taut} A$.

This means that for all values that WE assign to all the $\mathbf{p}, \mathbf{q}, \ldots$ that occur in A, and for all values that WE assign to all prime formulas —these behave as Boolean variables— we get the truth value of A come out TRUE.

Well, when we interpret A in some Interpretation $\mathfrak{D}$ we actually COMPUTE the values of the prime formulas in this interpretation (rather than assign them).

However, **the first paragraph above makes clear**, that whether we *COMPUTE* OR **ARBITRARILY ASSIGN** values to *the prime formulas* of A, the final value will be TRUE.

▶ A tautology does NOT CARE how the values of its variables are obtained!◀

So, $\models_{\mathfrak{D}} A$. As $\mathfrak{D}$ was arbitrary, I got

$$\models A$$

Valid Axioms 2. Ax2. $(\forall \mathbf{x})A \to A[\mathbf{x} := t]$ is valid.

Indeed, take a $\mathfrak{D}$, for the language of $A, \mathbf{x}, t$.

Now $\Big((\forall \mathbf{x})A \to A[\mathbf{x} := t]\Big)^{\mathfrak{D}}$ is

$$(\forall \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}} \to \Big(A[\mathbf{x} := t]\Big)^{\mathfrak{D}} \qquad (1)$$

To the left of $\to$ we explained the translation of $(\forall \mathbf{x})A$ in Remark 0.1.4 of the previous PDF, p.23).

Let's make the rhs of $\to$ more useable:

Claim: It is the same as $A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := t^{\mathfrak{D}}]$.

Indeed, start with the wff depicted as a box below.

$$A: \quad \boxed{\ldots \boxed{\mathbf{x}} \ldots \boxed{\mathbf{x}} \ldots}$$

Thus

$$A[\mathbf{x} := t]: \quad \boxed{\ldots \boxed{t} \ldots \boxed{t} \ldots} \tag{3}$$

Hence

$$\Big(A[\mathbf{x} := t]\Big)^{\mathfrak{D}}: \quad \boxed{(\ldots)^{\mathfrak{D}} \boxed{t^{\mathfrak{D}}} (\ldots)^{\mathfrak{D}} \boxed{t^{\mathfrak{D}}} (\ldots)^{\mathfrak{D}}} \tag{4}$$

But (4) is the result of applying "$[\mathbf{x} := t^{\mathfrak{D}}]$" to

$$A^{\mathfrak{D}}_{\mathbf{x}}: \quad (\ldots)^{\mathfrak{D}} \boxed{\mathbf{x}} (\ldots)^{\mathfrak{D}} \boxed{\mathbf{x}} (\ldots)^{\mathfrak{D}}$$

that is, it is the same as

$$A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := t^{\mathfrak{D}}]$$

With *the claim verified*, (1) is now TRUE:

Here is why: Assume the lhs of $\rightarrow$ in (1). That is, suppose $A^{\mathfrak{D}}_i$ is true for all $i \in D$. But then it is true IN PARTICULAR for $i = t^{\mathfrak{D}}$.

Valid Axioms 3. **Ax6**. $t = s \to (A[\mathbf{x} := t] \equiv A[\mathbf{x} := s])$. The translation of this in $\mathfrak{D}$ is —see the work we did for **Ax2**!)

$$t^{\mathfrak{D}} = s^{\mathfrak{D}} \to \\ (A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := t^{\mathfrak{D}}] \equiv A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := s^{\mathfrak{D}}]) \tag{1}$$

Assume the lhs of "$\to$" in (1). Thus $t^{\mathfrak{D}} = s^{\mathfrak{D}} = k \in D$.

The rhs of (1) becomes

$$A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := k] \equiv A^{\mathfrak{D}}_{\mathbf{x}}[\mathbf{x} := k]$$

which is trivially true.

Valid Axioms 4. For the remaining axioms there is nothing new to learn; see the text for proofs of their validity. Incidentally, the axiom $\mathbf{x} = \mathbf{x}$ has already been shown to be valid (0.1.2).

0.1.3 Metatheorem. (Soundness of Predicate Logic) *If* $\vdash A$, *then* $\models A$.

We omit the trivial proof by induction on proof length (we saw two such proofs).

For length one we the ONLY formula that appears in the proof is an axiom. But that is valid!

The induction step notes that our *ONLY PRIMARY*[†] *rule* preserves truth.

[†] Given up in front.

0.1.4 Example. (Strong Gen; Again!) Can our logic **prove** *strong generalisation* as a "derived rule"?

Namely, can we have

If $\Gamma \vdash A$, then $\Gamma \vdash (\forall \mathbf{x})A$, ***with NO restriction on*** $\mathbf{x}$?

If yes, take $\Gamma = \{A\}$.[†] We get

$$A \vdash (\forall \mathbf{x})A \tag{1}$$

By the DThm, (1) allows this:

$$\vdash A \to (\forall \mathbf{x})A \tag{2}$$

Soundness OBJECTS to (2):

If we got (2) then, by Soundness, we get

$$\models A \to (\forall \mathbf{x})A \tag{3}$$

I will contradict (3) showing

$$\not\models A \to (\forall \mathbf{x})A \tag{4}$$

The Definition of "$\models$" (0.1.1) dictates that I find ONE $\mathfrak{D}$ such that

$$(A \to (\forall \mathbf{x})A)^{\mathfrak{D}} = \mathbf{f} \tag{5}$$

This $\mathfrak{D}$ is called a countermodel of (2).

[†] Then $A \vdash A$, hence $A \vdash (\forall \mathbf{x})A$.

It is hopeless to search for a $\mathfrak{D}$ FOR A GENERAL A.

For a *countermodel* I ONLY need a SPECIFIC A (a countermodel is a counterexample!)

▶ Always work with an *atomic* formula in place of A.

Now then! If we have (3) *IN GENERAL*, THEN *we also have it for A being atomic*, in fact taking A to be "$x = y$" (3) should work!

DOES IT?

NO. We saw in Example 0.0.1(2.) (cf. Definition 0.1.1)

$$\not\models x = y \to (\forall x)x = y$$

So (2) is wrong and so is (1). $\square$

0.1.5 Example. (Important!) In the elimination of $\exists$ we start with $(\exists \mathbf{x})A$ (hypothesis, or proved from some Γ).

Then we add the associated Auxiliary Hypothesis

$$A[\mathbf{x} := \mathbf{z}]$$

where $\mathbf{z}$ is fresh for $(\exists \mathbf{x})A$ and for some other formulas.

Hypothesis? YES! Some folks think it is a conclusion of $(\exists \mathbf{x})A$.

Are they justified?

NO

.

Suppose this is a theorem schema

$$(\exists \mathbf{x})A \vdash A[\mathbf{x} := \mathbf{z}] \tag{1}$$

Then so is

$$\vdash (\exists \mathbf{x})A \to A[\mathbf{x} := \mathbf{z}] \tag{2}$$

by an application of DThm.

I will show that that the wff in (2) has a countermodel and *thus is not a theorem*. So, nor is (1).

As always start with an atomic special case of A to work with!

So if I can prove (2) then I can also prove

$$\vdash (\exists \mathbf{x})\mathbf{x} = \mathbf{y} \to \mathbf{z} = \mathbf{y}, \quad \mathbf{z} \text{ fresh} \tag{3}$$

Take $D = \mathbb{N}$ and $\mathbf{y}^{\mathfrak{D}} = 3$, $\mathbf{z}^{\mathfrak{D}} = 5$.

The formula in (3) translates as

$$\overbrace{(\exists \mathbf{x} \in \mathbb{N})\mathbf{x} = 3}^{\mathbf{t}} \to \overbrace{5 = 3}^{\mathbf{f}}$$

Thus $\not\models (\exists \mathbf{x})\mathbf{x} = \mathbf{y} \to \mathbf{z} = \mathbf{y}$ and *a fortiori*

$$\not\models (\exists \mathbf{x})A \to A[\mathbf{x} := \mathbf{z}]$$

By Soundness, (2) and hence ALSO (1) are **false** statements.

$\square$

0.1.6 Example. We have proved in class/NOTES/Text

$$\vdash (\exists \mathbf{y})(\forall \mathbf{x})A \rightarrow (\forall \mathbf{x})(\exists \mathbf{y})A$$

We hinted in class that we cannot also prove

$$\vdash (\forall \mathbf{x})(\exists \mathbf{y})A \rightarrow (\exists \mathbf{y})(\forall \mathbf{x})A \tag{1}$$

To show that (1) is unprovable I pick a countermodel (=an interpretation that makes the wff in it false).

Pick A to be something simple. Atomic is best!

I take $D = \mathbb{N}$ and $\mathbf{x} = \mathbf{y}$ for A. Translating the wff in (1) I note

$$\overbrace{(\forall \mathbf{x} \in \mathbb{N})(\exists \mathbf{y} \in \mathbb{N})\mathbf{x} = \mathbf{y}}^{\mathbf{t}} \rightarrow \overbrace{(\exists \mathbf{y} \in \mathbb{N})(\forall \mathbf{x} \in \mathbb{N})\mathbf{x} = \mathbf{y}}^{\mathbf{f}}$$

Since the interpretation falsifies a special case of (1) the latter is not provable (by soundness). □

0.1.7 Example. We noted in class/NOTES/Text that we cannot prove

$$\vdash (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \to (\exists \mathbf{x})(A \wedge B) \tag{1}$$

To demonstrate this fact now we use Soundness and countermodels.

So, I pick a countermodel.

Pick A and B to be something simple. Atomic is best!

I take $D = \mathbb{N}$ and "$\mathbf{x}$ is even" for A while I take "$\mathbf{x}$ is odd" for B. Translating the wff in (1) I note

$$\overbrace{(\exists \mathbf{x} \in \mathbb{N})\mathbf{x} \text{ is even} \wedge (\exists \mathbf{x} \in \mathbb{N})\mathbf{x} \text{ is odd}}^{\mathbf{t}} \to \overbrace{(\exists \mathbf{x} \in \mathbb{N})(\mathbf{x} \text{ is even} \wedge \mathbf{x} \text{ is odd})}^{\mathbf{f}}$$

Since the interpretation falsifies a special case of (1) the latter is not provable (by soundness). $\square$

0.1.8 Exercise. On the other hand, do prove by $\exists$-elimination the other direction: We DO have

$$\vdash (\exists \mathbf{x})(A \wedge B) \rightarrow (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B$$

$\square$

0.1.9 Example. (Important!) Why is $D \neq \emptyset$ important?

Well let us start by proving

$$\vdash (\forall \mathbf{x})A \to (\exists \mathbf{x})A \tag{1}$$

Use DThm to prove instead

$$(\forall \mathbf{x})A \vdash (\exists \mathbf{x})A$$

1) $(\forall \mathbf{x})A$ $\langle$hyp$\rangle$
2) A $\langle 1 +$ spec$\rangle$
3) $(\exists \mathbf{x})A$ $\langle 2 +$ Dual spec$\rangle$

However, if I took $\mathfrak{D} = (D, M)$ with $D = \emptyset$ then look at the transaltion of the formula in (1):

$$\overbrace{(\forall \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}}^{\textbf{t vacuously}}{}^{*} \to \overbrace{(\exists \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}}^{\textbf{f}} \tag{2}$$

Soundness fails for the formula in (1). *We DON'T like this! So we NEVER allow $D = \emptyset$.* $\square$

*Do not forget that "$(\forall \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}$" means "$(\forall \mathbf{x})(\mathbf{x} \in D \to A^{\mathfrak{D}}_{\mathbf{x}})$", while "$(\exists \mathbf{x} \in D)A^{\mathfrak{D}}_{\mathbf{x}}$" means "$(\exists \mathbf{x})(\mathbf{x} \in D \wedge A^{\mathfrak{D}}_{\mathbf{x}})$".