This note is about the \textit{Soundness} and \textit{Completeness} ("Post’s Theorem") in Boolean logic.

\section*{1. Soundness of Boolean Logic}

\textit{Soundness} is the Property expressed by the statement of the metatheory below:

\begin{equation}
\text{If } \Gamma \vdash A, \text{ then } \Gamma \models_{\text{taut}} A \tag{1}
\end{equation}

\subsection*{1.1 Definition.} The statement "Boolean logic is Sound" means that Boolean logic satisfies (1).  \hfill \square
To prove soundness is an easy induction on the length of $\Gamma$-proofs:

We prove that proofs preserve truth.
1.2 Lemma. Eqn and Leib preserve truth, that is,

\[ A, A \equiv B \models_{\text{taut}} B \]  \hspace{1cm} (2)

and

\[ A \equiv B \models_{\text{taut}} C[p : A] \equiv C[p := B] \]  \hspace{1cm} (3)

Proof. We proved (2) in Assignment #1. We prove (2) now:

So, let a state \( s \) make \( A \equiv B \) true (t).

We will show that

\[ C[p := A] \equiv C[p := B] \] is t in state \( s \) \hspace{1cm} (4)

If \( p \) is not in \( C \) then (4) is \( C \equiv C \), a tautology, so is true under \( s \) in particular.

\[ \text{Let then the distinct } p, q, r, r', r'', \ldots \text{ all occur in } C. \]

Now in the lhs of (4) \( p \) gets the value \( \overline{s}(A) \), while \( q, r, r', r'', \ldots \) get their values DIRECTLY from \( s \).

Now, in the RHS of (4) \( p \) gets the value \( \overline{s}(B) \), while \( q, r, r', r'', \ldots \) STILL get their values DIRECTLY from \( s \).

\[ \text{But } \overline{s}(A) = \overline{s}(B). \]

So both the lhs and rhs of (4) end up with the same truth value after the indicated substitutions.

In short, the equivalence is true.  \( \square \)
We can now prove:

1.3 Metatheorem. Boolean logic is Sound, that is, (1) on p.1 holds.

Proof. By induction on the length of proof, $n$, needed to obtain $\Gamma \vdash A$ we prove

$$\Gamma \models taut A$$  \hfill (†)

So pick a state $s$ that satisfies $\Gamma$. \hfill (‡)

1. **Basis.** $n = 1$. Then we have just $A$ in the proof.

   If $A \in \Lambda$, then it is a tautology, so in particular is true under $s$. We have (†).

   If $A \in \Gamma$, then $s$ satisfies $A$. Again we have (†).

   **I.H.** Assume for all proofs of length $\leq n$.

   **I.S.** Prove the theorem in the case (†) needed a proof of length $n + 1$:

$$\begin{align*}
\text{length } &= n \\
\underbrace{\ldots} & , A \\
\text{length } &= n+1
\end{align*}$$

Now if $A$ is in $\Lambda \cup \Gamma$ we are back to the Basis. Done.
If not

- Case where \( A \) is the result of \( EQN \) on \( X \) and \( X \equiv Y \) that are in the “...-area”.
  By the I.H. \( s \) satisfies \( X \) and \( X \equiv Y \) hence, by the Lemma, satisfies \( A \).

- Case where \( A \) is the result of \( LEIB \) on \( X \equiv Y \) that are in the “...-area”.
  By the I.H. \( s \) satisfies \( X \equiv Y \) hence, by the Lemma, satisfies \( A \).

\[ \square \]

1.4 Corollary. \textit{If} \( \vdash A \) \textit{then} \( \models_{\text{taut}} A \). \textit{A is a tautology.}

\textit{Proof.} \( \Gamma = \emptyset \) here. BUT, EVERY state \( s \) satisfies THIS \( \Gamma \), vacuously:

Indeed, to prove that some state \( v \) does NOT you \textit{need} an \( X \in \emptyset \) such that \( \overline{v}(X) = f \); \textit{impossible}.

Hence every state satisfies \( A \). Thus a \( \models_{\text{taut}} A \). \[ \square \]

1.5 Example. Soundness allows us to \textit{disprove} formulas: To show they are NOT theorems.

- \( \vdash p \) is false. If this were true, then \( p \) would be a tautology!

- \( \vdash \bot \) is false! Because \( \bot \) is not a tautology!

- \( p \vdash p \land q \) is false. Because if it were true I’d have to have \( p \models_{\text{taut}} p \land q \).
  \textit{Not so:} Take a state \( s \) such that \( s(p) = t \) and \( s(q) = f \). \[ \square \]
2. Completeness of Boolean logic
   (“Post’s Theorem”)

We prove here

(1) A weak form of Post’s theorem: If \( \Gamma \) is finite and \( \Gamma \models_{\text{taut}} A \), then \( \Gamma \vdash A \)
   and derive as a corollary the **Deduction Theorem**:

(2) If \( \Gamma, A \vdash B \), then \( \Gamma \vdash A \rightarrow B \).
2. Completeness of Boolean logic ("Post's Theorem")

2.1. Some tools

We will employ the TWO results from class/text below:

2.1 Theorem. \( \vdash \neg (A \lor B) \lor C \equiv (\neg A \lor C) \land (\neg B \lor C) \).

2.2 Theorem. \( \neg A \lor C, \neg B \lor C \vdash \neg (A \lor B) \lor C \).

2.3 Main Lemma. Suppose that \( A \) contains none of the symbols \( \top, \bot, \rightarrow, \land, \equiv \). If \( \models_{\text{taut}} A \), then \( \vdash A \).

Proof. The proof is long but easy!

Under the assumption, \( A \) is an \( \lor \)-chain, that is, it has the form

\[ A_1 \lor A_2 \lor A_3 \lor \ldots \lor A_i \lor \ldots \lor A_n \]  

(1)

where none of the \( A_i \) has the form \( B \lor C \).

In (1) we assume without loss of generality that \( n > 1 \), due to the axiom \( X \lor X \equiv X \) — that is, in the contrary case we can use \( A \lor A \) instead, which is a tautology as well.

Moreover, (1), that is \( A \), is written in least parenthesised notation.

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I. A Weak Post’s Theorem and the Deduction Theorem Retold

Let us call an $A_i$ reducible iff it has the form $\neg(C \lor D)$ or $\neg(\neg C)$.

“Reducible”, since $A_i$ is not alone in the $\lor$-chain, will be synonymous to simplifiable.

Otherwise $A_i$ is irreducible. Not simplifiable.

Thus, the only possible irreducible $A_i$ have the form $p$ or $\neg p$ (where $p$ is a variable).

We say that $p$ “occurs positively in ... $\lor p \lor ...$”, while it “occurs negatively in ... $\lor \neg p \lor ...$”.

In, for example, $p \lor \neg p$ it occurs both positively and negatively.

**By definition we will say that $A$ is irreducible iff all its $A_i$ are.**

We define the reducibility degree, of EACH $A_i$ —in symbols, $rd(A_i)$— to be the total number, counting repetitions of the $\neg$ and $\lor$ connectives in it, **not counting a possible leading $\neg$.**
The reducibility degree of the entire \( A \) is the sum of the reducibility degrees of all its \( A_i \).

For example, \( rd(p) = 0, \: rd(\neg p) = 0, \: rd(\neg(\neg p \lor q)) = 2, \: rd(\neg(\neg p \lor \neg q)) = 3 \), \( rd(\neg p \lor q) = 0 \).

By induction on \( rd(A) \) we now prove the main lemma, that \( \vdash A \) on the stated hypothesis that \( \models_{\text{taut}} A \).

For the Basis, let \( A \) be an irreducible tautology —so, \( rd(A) = 0 \).

It must be that \( A \) is a string of the form

“\( \cdots \lor p \lor \cdots \neg p \lor \cdots \)”

for some \( p \), otherwise,

if no \( p \) appears both “positively” and “negatively”,

then we can find a truth-assignment that makes \( A \) false \((f)\) —a contradiction to its tautologyhood.

To see that we can do this, just assign \( f \) to \( p \)’s that occur positively only, and \( t \) to those that occur negatively only.
Now

\[ A \Leftrightarrow \langle \text{commuting the terms of an } \lor\text{-chain} \rangle \]

\[ p \lor \neg p \lor B \quad \text{(what is “B”?)} \]

\[ \Leftrightarrow \langle \text{Leib + axiom + Red. } \top \text{ META; Denom: } r \lor B; \text{ fresh } r \rangle \]

\[ \top \lor B \quad \text{bingo!} \]

Thus \( \vdash A \), which settles the Basis-case: \( rd(A) = 0 \).

We now argue the case where \( rd(A) = m + 1 \), on the I.H. that for any formula \( Q \) with \( rd(Q) \leq m \), we have that \( \models_{\text{taut}} Q \) implies \( \vdash Q \).

By commutativity (symmetry) of “\( \lor \)”, let us assume without restricting generality that \( rd(A_1) > 0 \).

We have two cases:

1. \( A_1 \) is the string \( \neg \neg C \), hence \( A \) has the form \( \neg \neg C \lor D \).

Clearly \( \models_{\text{taut}} C \lor D \) as well.

Moreover, \( rd(C \lor D) < rd(\neg \neg C \lor D) \), hence (by I.H.)

\[ \vdash C \lor D \quad (†) \]
But,

$$\neg\neg C \lor D$$

$$\iff \langle \text{Leib } + \vdash \neg\neg X \equiv X; \text{ Denom: } r \lor D; \text{ fresh } r \rangle$$

$$C \lor D \quad (†) \text{ above is "bingo"!}$$

Hence, $$\vdash \neg\neg C \lor D$$, that is, $$\vdash A$$ in this case.

Lecture #12, Oct. 23

One more case to go:

(2) $$A_1$$ is the string $$\neg(C \lor D)$$, hence $$A$$ has the form

$$\neg(C \lor D) \lor E.$$

We want: $$\vdash \neg(C \lor D) \lor E$$   (i)

By 2.1 and from $$\models_{\text{taut}} \neg(C \lor D) \lor E$$ — this says $$\models_{\text{taut}} A$$— we immediately get that

$$\models_{\text{taut}} \neg C \lor E$$   (ii)

and

$$\models_{\text{taut}} \neg D \lor E$$   (iii)

from the $$\equiv$$ and $$\land$$ truth tables.

Since the $$rd$$ of each of (ii) and (iii) is $$< rd(A)$$, the I.H. yields $$\vdash \neg C \lor E \text{ AND } \vdash \neg D \lor E.$$
Apply the RULE 2.2 to the above two theorems to get (i).
We are done, **except for one small detail:**

If we had changed the “original” $A$ into $A \lor A$ (cf. the “without loss of generality” remark just below (1) on p.7), then all we proved above is $\vdash A \lor A$.

The *contraction* rule from (e-)Class, Notes, and Text then yield $\vdash A$. $\square$

But ALL this only proves “$|=_{\text{taut}} A$ implies $\vdash A$” when $A$ does not contain $\land, \rightarrow, \equiv, \bot, \top$.

**WHAT IF IT DOES?**
We are now removing the restriction on $A$ regarding its connectives and constants:

**2.4 Metatheorem. (Post’s Theorem)** If $\models_{\text{taut}} A$, then $\vdash A$.

*Proof.* First, we note the following *theorems* stating equivalences, *where* $p$ *is fresh for* $A$.

The proof of the last one is in the notes and text but it was too long (but easy) to cover in class.

\[
\begin{align*}
\vdash \top &\equiv \neg p \lor p \\
\vdash \bot &\equiv \neg (\neg p \lor p) \\
\vdash C \rightarrow D &\equiv \neg C \lor D \\
\vdash C \land D &\equiv \neg (\neg C \lor \neg D) \\
\vdash (C \equiv D) &\equiv ((C \rightarrow D) \land (D \rightarrow C))
\end{align*}
\]

Using (2) above, we **eliminate**, **in order**, all the $\equiv$, then all the $\land$, then all the $\rightarrow$ and finally all the $\bot$ and all the $\top$. 

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* A “Weak” Post’s Theorem and the Deduction Theorem © by George Tourlakis
Let us assume that our process eliminates one unwanted symbol at a time.

This leads to the Equational Proof below.

Starting from $A$ we will generate a sequence of formulae
\[ F_1, F_2, F_3, \ldots, F_n \]
where $F_n$ contains no $\top, \bot, \land, \rightarrow, \equiv$.

I am using here $F_1$ as an alias for $A$. We will also give to $F_n$ an alias $A'$. 

\[ A \iff \langle \text{Leib from (2)} \rangle \]
\[ F_2 \]
\[ \iff \langle \text{Leib from (2)} \rangle \]
\[ F_3 \]
\[ \iff \langle \text{Leib from (2)} \rangle \]
\[ F_4 \]
\[ \vdots \]
\[ \iff \langle \text{Leib from (2)} \rangle \]
\[ A' \]
Thus, \( \vdash A' \equiv A \) \hfill (*)

By **soundness**, we also have \( \models_{taut} A' \equiv A \) \hfill (**) 

So, say \( \models_{taut} A \). By (**) we have \( \models_{taut} A' \) as well, and by the Main Lemma 2.3 we obtain \( \vdash A' \).

By (*) and Eqn we get \( \vdash A \). \hfill \square

Post’s theorem is the “*Completeness Theorem*”\(^\dagger\) of Boolean Logic.

It shows that the syntactic manipulation apparatus — proofs — DOES certify the “whole truth” (tautologyhood) in the Boolean case.

\(^\dagger\)Which is really a *Meta*theorem, right?

\( \dagger \) Which is really a Metatheorem, right?
2.5 Corollary. If $A_1, \ldots, A_n \models taut B$, then $A_1, \ldots, A_n \vdash B$.

Proof. It is an easy semantic exercise to see (see the special case in Problem Set #1, Fall 2020) that

$$\models_{taut} A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$$

By 2.4,

$$\vdash A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$$

hence (hypothesis strengthening)

$$A_1, \ldots, A_n \vdash A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n \rightarrow B \quad (1)$$

Applying modus ponens $n$ times to (1) we get

$$A_1, \ldots, A_n \vdash B$$

\[ \square \]

The above corollary is very convenient.

It says that every (correct) schema $A_1, \ldots, A_n \models taut B$ leads to a derived rule of inference, $A_1, \ldots, A_n \vdash B$.

In particular, combining with the transitivity of $\vdash$ metatheorem, we get

2.6 Corollary. If $\Gamma \vdash A_i$, for $i = 1, \ldots, n$, and if $A_1, \ldots, A_n \models taut B$, then $\Gamma \vdash B$. 
Thus—unless otherwise requested!—we can, from now on, rigorously mix syntactic with semantic justifications of our proof steps.

For example, we have at once $A \land B \vdash A$, because (trivially) $A \land B \models_{\text{taut}} A$ (compare with our earlier, much longer, proof given in class).
3. Deduction Theorem, Proof by Contradiction

3.1 Metatheorem. (The Deduction Theorem) If \( \Gamma, A \vdash B \), then \( \Gamma \vdash A \rightarrow B \), where \( \Gamma, A \) means “all the assumptions in \( \Gamma \), plus the assumption \( A \)” (in set notation this would be \( \Gamma \cup \{A\} \)).

Proof. Let \( G_1, \ldots, G_n \subseteq \Gamma \) be a finite set of formulae used in a \( \Gamma, A \)-proof of \( B \).

Thus we also have \( G_1, \ldots, G_n, A \vdash B \).

By soundness, \( G_1, \ldots, G_n, A \models_{\text{taut}} B \).

But then,

\[
G_1, \ldots, G_n \models_{\text{taut}} A \rightarrow B
\]

By 2.5, \( G_1, \ldots, G_n \vdash A \rightarrow B \) and hence \( \Gamma \vdash A \rightarrow B \) by hypothesis strengthening. \( \square \)
The mathematician, or indeed the mathematics practitioner, uses the Deduction theorem all the time, without stopping to think about it. Metatheorem 3.1 above makes an honest person of such a mathematician or practitioner.

The everyday “style” of applying the Metatheorem goes like this:

Say we have all sorts of assumptions and we want, under these assumptions, to “prove” that “if \( A \), then \( B \)” (verbose form of “\( A \to B \)”).

We start by adding \( A \) to our assumptions, often with the words, “Assume \( A \)”. We then proceed and prove just \( B \) (not \( A \to B \)), and at that point we rest our case.

Thus, we may view an application of the Deduction theorem as a simplification of the proof-task. It allows us to “split” an implication \( A \to B \) that we want to prove, moving its premise to join our other assumptions. We now have to prove a simpler formula, \( B \), with the help of stronger assumptions (that is, all we knew so far, plus \( A \)). That often makes our task so much easier!
An Example. Prove
\[
\vdash (A \rightarrow B) \rightarrow A \lor C \rightarrow B \lor C
\]

By DThm, suffices to prove
\[
A \rightarrow B \vdash A \lor C \rightarrow B \lor C
\]

instead.

Again By DThm, suffices to prove
\[
A \rightarrow B, A \lor C \vdash B \lor C
\]

instead.

Let’s do it:

1. \(A \rightarrow B\) \hspace{1cm} \langle \text{hyp} \rangle
2. \(A \lor C\) \hspace{1cm} \langle \text{hyp} \rangle
3. \(A \rightarrow B \equiv \neg A \lor B\) \hspace{1cm} \langle \neg \lor \text{thm} \rangle
4. \(\neg A \lor B\) \hspace{1cm} \langle 1 + 3 + \text{Eqn} \rangle
5. \(B \lor C\) \hspace{1cm} \langle 2 + 4 + \text{Cut} \rangle

\[ \square \]
3.2 Definition. A set of formulas $\Gamma$ is *inconsistent* or *contradictory* iff $\Gamma$ proves every $A$ in WFF.

Why "contradictory"? For if $\Gamma$ proves everything, then it also proves $p \land \lnot p$.

3.3 Lemma. $\Gamma$ is inconsistent iff $\Gamma \vdash \bot$

*Proof.* *only if*-part. If $\Gamma$ is as in 3.2, then, in particular, it proves $\bot$ since the latter is a well formed formula.

*if*-part. Say, conversely, that we have

$$\Gamma \vdash \bot \quad (9)$$

Let now $A$ be any formula in WFF whatsoever. We have

$$\bot \models_{taut} A \quad (10)$$

**Pause.** Do you believe (10)?

By Corollary 3.4, $\Gamma \vdash A$ follows from (9) and (10).

3.4 Metatheorem. *(Proof by contradiction)* $\Gamma \vdash A$ iff $\Gamma \cup \{\lnot A\}$ is inconsistent.

*Proof.* *if*-part. So let (by 3.3)

$$\Gamma, \lnot A \vdash \bot$$

Hence

$$\Gamma \vdash \lnot A \rightarrow \bot \quad (1)$$

by the Deduction theorem. However $\lnot A \rightarrow \bot \models_{taut} A$, hence, by Corollary 2.6 and (1) above, $\Gamma \vdash A$.

*only if*-part. So let

$$\Gamma \vdash A$$
By hypothesis strengthening,
\[ \Gamma, \neg A \vdash A \]  
(2)

Moreover, trivially,
\[ \Gamma, \neg A \vdash \neg A \]  
(3)

Since \( A, \neg A \models_{\text{taut}} \bot \), (2) and (3) yield \( \Gamma, \neg A \vdash \bot \) via Corollary 2.6, and we are done by 3.3. \( \square \)

3.4 legitimises the tool of “proof by contradiction” that goes all the way back to the ancient Greek mathematicians: To prove \( A \) assume instead the “opposite”, \( \neg A \). Proceed then to obtain a contradiction. This being accomplished, it is as good as having proved \( A \). \( \diamond \)