York University
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MATH 1090A. MID TERM, October 25, 2023; SOLUTIONS
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Question 1. (3 MARKS) Prove that NO wff is the empty string λ by showing it contains AT LEAST ONE symbol from the Boolean alphabet.

The proof must be by analysing either formula constructions, or by the recursive definition of formulas.

Proof.

(a) **By formula construction/calculation.**

Well, a wff is a string that *appears* by itself on a line of a formula calculation.

What do we legally write on such a line? ONE of the following:

- An atomic wff, such as $\bot, \top, \mathbf{p}$. By inspection, none of these is λ.
- A string $(\neg B)$ licensed by the fact that we wrote the string B earlier. But this contains, for example $\neg$ —also "(" and ")"— so it too is not λ.
- A string $(B \circ C)$ —where $\circ \in \{\wedge, \vee, \rightarrow, \equiv\}$— licensed by the fact that we wrote the strings B *and* C earlier. But this string, this wff, contains, for example $\circ$ —but also "(" and ")"— so it too is not λ.

So, none of the strings I may EVER write in a step of a formula calculation can be λ. THUS, no wff can be λ as a wff is *precisely* a string *I may write* in a step of a formula calculation! In short, no wff can be λ

(b) **By recursive formula definition.**

Show that any wff A is $\neq \lambda$.

Well, A is one of the following three:

- $\bot, \top, \mathbf{p}$. Each of them $\neq \lambda$.
- $(\neg B)$. NOT λ! Contains, say, "$\neg$" (no quotes).
- $(B \circ C)$. NOT λ! Contains, say, "$\circ$" (no quotes). □

Question 2. (4 MARKS) Give an **Equational proof** of $\vdash A \to B \equiv A \land B \equiv A$.

Any other proof will max 0 marks.

Proof.

$$
\begin{aligned}
& A \to B \\
\Leftrightarrow \quad & \langle \text{axiom} \rangle \\
& A \lor B \equiv B \\
\Leftrightarrow \quad & \langle \text{GR axiom} \rangle \\
& A \land B \equiv A
\end{aligned}
$$

□

Question 3. (5 MARKS) Give an **Equational proof** of the following:

$$\vdash A \lor B \equiv (A \to \bot) \to B$$

Any other proof maxes to zero!

Proof.

$A \lor B$
$\Leftrightarrow$ ⟨Leib+double neg; Denom: $\mathbf{p} \lor B$⟩
$\neg\neg A \lor B$
$\Leftrightarrow$ ⟨$\neg\lor$-thm⟩
$\neg A \to B$
$\Leftrightarrow$ ⟨Leib+thm; Denom: $\mathbf{p} \to B$⟩
$(\neg A \lor \bot) \to B$ **Comment**: this "obvious" step I had omitted; I should get "3/5" in my old proof! :)
$\Leftrightarrow$ ⟨Leib+$\neg\lor$-thm; Denom: $\mathbf{p} \to B$⟩
$(A \to \bot) \to B$

□

Question 4. (a) (4 MARKS) For **any** A and B *prove* the following via an **Equational Proof**.

$$\vdash A \wedge \neg A \rightarrow B$$

Any other proof maxes to a zero.

Proof.

$A \wedge \neg A \rightarrow B$
$\Leftrightarrow$ $\langle \neg\vee$-thm$\rangle$
$\neg(A \wedge \neg A) \vee B$
$\Leftrightarrow$ $\langle$deMorgan + Leib; Denom: $\neg \mathbf{p} \vee B\rangle$
$\neg\neg(\neg A \vee A) \vee B$
$\Leftrightarrow$ $\langle$double neg + Leib; Denom: $\mathbf{p} \vee B\rangle$
$(\neg A \vee A) \vee B$
$\Leftrightarrow$ $\langle$Red. $\top$ METAthm + axiom + Leib; Denom: $\mathbf{p} \vee B\rangle$
$\top \vee B$ bingo!

□

(b) (2 MARKS) Use a **Hilbert style proof** AND the above result to prove

$$A \wedge \neg A \vdash B$$

Any other proof maxes to a zero.
Proof.

(1)	$A \wedge \neg A$	$\langle$hyp$\rangle$
(2)	$A \wedge \neg A \rightarrow B$	$\langle$thm in subquestion (a)$\rangle$
(3)	B	$\langle$1 + 2 + *modus ponens* (MP)$\rangle$

□

Extra blank “overflow” page for answers