Lassonde School of Engineering

Dept. of EECS
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MATH1090 A. Problem Set No. 3 —SOLUTIONS
Posted: Nov. 24, 2023

Unless a required proof style (e.g., by resolution, Equational, Hilbert) is used in your answer, then your answer is graded out of 0.

Exercise #5 has had a typo that was corrected only yesterday. So, it is NOT required.

(5 POINTS Max for each question) **Do all of the following:**

All resolution proofs below MUST use the graphical technique. Minimise preprocessing. You lose marks if your preprocessing is so long that it solves the problem WITHOUT doing any resolution step.

1. Use Resolution to prove $\vdash A \rightarrow \neg(\neg A \wedge \neg B)$.

Proof. Resolution MUST go via proof-by-contradiction, so

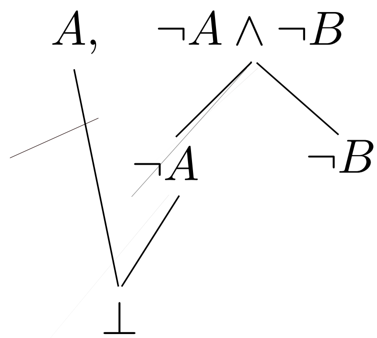
(a) Use DThm first to prove instead

$$A \vdash \neg(\neg A \wedge \neg B)$$

(b) Prove instead (**by contradiction**)

$$A, \neg A \wedge \neg B \vdash \bot$$

Here is the proof:



$\square$

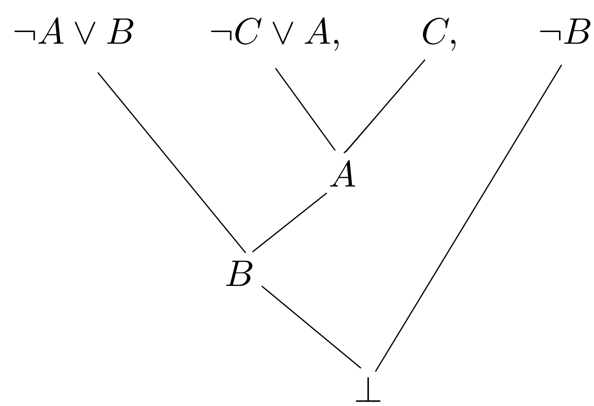
2. Use Resolution to prove, for any A, B, C, that $\vdash (A \to B) \to (C \to A) \to (C \to B)$.

Proof. By DThm (3 times) it suffices to prove the following instead:

$$(A \to B), (C \to A), C \vdash B$$

By proof by contradiction do instead (I directly applied $\neg\vee$)

$$\neg A \vee B, \neg C \vee A, C, \neg B \vdash \bot$$



$\square$

3. Use Resolution to prove, for any A, B, C, D, that

$$\vdash (A \vee B \vee C) \wedge (A \to D) \wedge (B \to D) \wedge (C \to D) \to D$$

by DThm, prove instead ($\neg\vee$ applied)

$$(A \vee B \vee C) \wedge (\neg A \vee D) \wedge (\neg B \vee D) \wedge (\neg C \vee D) \vdash D$$

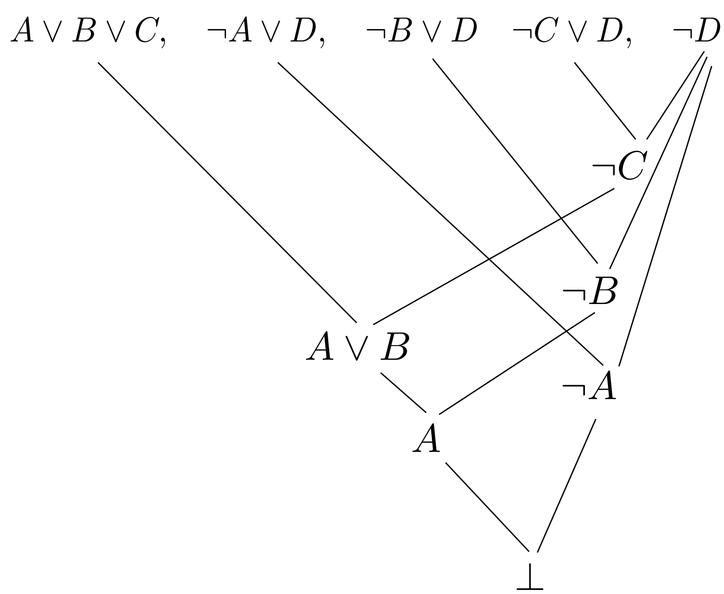
By hypothesis splitting we do instead

$$A \vee B \vee C, \quad \neg A \vee D, \quad \neg B \vee D, \quad \neg C \vee D \vdash D$$

By *proof by contradiction* do instead

$$A \vee B \vee C, \quad \neg A \vee D, \quad \neg B \vee D \quad \neg C \vee D, \quad \neg D \vdash \bot \tag{1}$$

So we prove (1):



□

4. You are in Boolean Logic.

Define: Σ is satisfiable, by definition, if some state s makes all the wff in Σ true. If no such s exists then we call Σ UNsatisfiable.

Prove that if a finite set of wff Σ is unsatisfiable, then $\Sigma \vdash \bot$.

Proof. Start with statement $(*)$ below for an unsatisfiable finite $(*)$:

$$\Sigma \models_{taut} \bot \qquad (*)$$

The statement $(*)$ is valid since there is **no counterexample** (a counterexample would require an s that satisfies Σ).

Then, by finiteness and Post's Theorem, $(*)$ implies $\Sigma \vdash \bot$ □

5. (**Removed**. Optionally you can do it to get extra points *if correct*) Prove that for any object variables $\mathbf{x}, \mathbf{y}, \mathbf{z}$ we have the absolute theorem $\vdash \mathbf{x} = \mathbf{y} \to (\mathbf{y}{=}\mathbf{z}) \to (\mathbf{x}{=}\mathbf{z})$.

Hint. Use a **Hilbert** style proof using the axioms of equality.

Proof as the Exercise typo was removed on Nov. 23 (see red "=" above).

Take as "A" the wff $\mathbf{w} = \mathbf{z}$ in **Ax6**. Then an **Axiom 6** instance is

$$\mathbf{x} = \mathbf{y} \to \Big((\mathbf{x}{=}\mathbf{z}) \equiv (\mathbf{y}{=}\mathbf{z})\Big)$$

so here is a simple Hilbert proof:

1) $\mathbf{x} = \mathbf{y} \to \Big((\mathbf{x}{=}\mathbf{z}) \equiv (\mathbf{y}{=}\mathbf{z})\Big)$ $\langle$Ax6$\rangle$
2) $\mathbf{x} = \mathbf{y} \to \Big((\mathbf{y}{=}\mathbf{z}) \to (\mathbf{x}{=}\mathbf{z})\Big)$ $\langle 1 + \text{Post}\rangle$

□

Do **NOT** use the Auxiliary Hypothesis Metatheorem in THIS Problem Set!

6. Prove that $\vdash A \to B$ implies $\vdash (\exists \mathbf{x})A \to (\exists \mathbf{x})B$.

Required Methodology. Use a **Hilbert** style proof and the metatheorem from class "$\vdash A \to B$ implies $\vdash (\forall \mathbf{x})A \to (\forall \mathbf{x})B$".

Proof.

1) $A \to B$ ⟨abs. thm⟩
2) $\neg B \to \neg A$ ⟨1 + Post⟩
3) $(\forall \mathbf{x})\neg B \to (\forall \mathbf{x})\neg A$ ⟨2 + A-MON (OK, $\Gamma = \emptyset$)⟩
4) $\neg(\forall \mathbf{x})\neg A \to \neg(\forall \mathbf{x})\neg B$ ⟨3 + Post⟩

Line (4) says "$(\exists \mathbf{x})A \to (\exists \mathbf{x})B$" using the abbreviation "$\exists$". □

7. Prove **Hilbert** style, that $\vdash (\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\exists \mathbf{x})B$.

Proof. We prove a **Lemma**: $\vdash Q \to (\exists \mathbf{x})Q$:

1) $(\forall \mathbf{x})\neg Q \to \neg Q$ ⟨Ax2⟩
2) $Q \to \neg(\forall \mathbf{x})\neg Q$ ⟨1 + Post⟩

But line 2) says what we want, by definition of $\exists$. **END of Lemma PROOF**

Main Proof Now.

1) $(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\forall \mathbf{x})B$ ⟨Ax4⟩
2) $(\forall \mathbf{x})B \to B$ ⟨Ax2⟩
3) $B \to (\exists \mathbf{x})B$ ⟨Lemma⟩
4) $(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\exists \mathbf{x})B$ ⟨(1, 2, 3) + Post⟩

END OF MAIN PROOF. □

8. Prove **Hilbert** style, that

$$\vdash (\forall \mathbf{x})(A \vee B \to C) \to (\forall \mathbf{x})(A \to C) \wedge (\forall \mathbf{x})(B \to C)$$

Proof. We have seen the wff below many times before in the Boolean Part of our lectures/Notes ("proof by cases" we called it).

$$\models_{taut} (A \vee B \to C) \equiv \Big((A \to C) \wedge (B \to C)\Big)$$

Trivially (think Ping-Pong!) we also have

$$\models_{taut} (A \vee B \to C) \to \Big((A \to C) \wedge (B \to C)\Big)$$

hence —since the above is in the **Ax1** group— we have:

$$\vdash (A \vee B \to C) \to (A \to C) \wedge (B \to C) \tag{1}$$

By A-MON applied to (1) we have

$$\vdash (\forall \mathbf{x})(A \vee B \to C) \to (\forall \mathbf{x})\Big((A \to C) \wedge (B \to C)\Big) \tag{2}$$

We conclude with a short Hilbert proof:

1) $(\forall \mathbf{x})(A \vee B \to C) \to (\forall \mathbf{x})\Big((A \to C) \wedge (B \to C)\Big)$ ⟨by (2) above⟩
2) $(\forall \mathbf{x})\Big((A \to C) \wedge (B \to C)\Big) \equiv (\forall \mathbf{x})(A \to C) \wedge (\forall \mathbf{x})(B \to C)$ ⟨by $\forall$ over $\wedge$⟩
3) $(\forall \mathbf{x})(A \vee B \to C) \to (\forall \mathbf{x})(A \to C) \wedge (\forall \mathbf{x})(B \to C)$ ⟨1 + 2 + Post⟩

□