Lassonde School of Engineering
EECS
MATH1090. Problem Set No. 4
Posted: Nov. 18, 2017

Due: Dec. 4, 2017, by 3:00pm; in the course assignment box.

It is worth remembering (from the course outline):

The homework must be each individual's own work. While consultations with the instructor, tutor, and among students, are part of the learning process and are encouraged, nevertheless, *at the end of all this consultation* each student will have to produce an individual report rather than a copy (full or partial) of somebody else's report.

The concept of "late assignments" does not exist in this course.

In what follows, "give a proof of $\vdash A$" means to give an equational or Hilbert-style proof of A, unless some other proof style is required (e.g., Resolution).

Annotation is always required!

*Do the following problems (**5 MARKS/Each**).*

In a number of problems that follow I use the notation of bounded quantifiers,

- $(\forall \mathbf{x})_A B$ —which is short for $(\forall \mathbf{x})(A \to B)$

  and

- $(\exists \mathbf{x})_A B$ —which is short for $(\exists \mathbf{x})(A \land B)$.

In each case I recommend that you expand the shorthand into standard notation **before** you proceed with your proofs, that is, e.g., $(\forall \mathbf{x})_A B$ is replaced by $(\forall \mathbf{x})(A \to B)$, etc.

1. Show that
$$\vdash (\forall \mathbf{x})_A(B \to C) \to (\forall \mathbf{x})_A B \to (\forall \mathbf{x})_A C$$

2. Show that
$$\vdash (\exists \mathbf{x})_{A \wedge B} C \equiv (\exists \mathbf{x})_A(B \wedge C)$$

3. Show that
$$\vdash (\exists \mathbf{x})A \to \Big((\exists \mathbf{x})_A(B \vee C) \equiv B \vee (\exists \mathbf{x})_A C\Big), \text{ where } \mathbf{x} \text{ is not free in } B$$

4. Show definitively **whether** omitting "$(\exists \mathbf{x})A \to$" above makes or not a difference to provability, that is, **prove** or **disprove**
$$\vdash (\exists \mathbf{x})_A(B \vee C) \equiv B \vee (\exists \mathbf{x})_A C, \text{ where } \mathbf{x} \text{ is not free in } B$$

5. Prove using **soundness**: $\nvdash (\forall \mathbf{y})(\exists \mathbf{x})A \to (\exists \mathbf{x})(\forall \mathbf{y})A$

6. Prove using **soundness**: $\nvdash (\exists \mathbf{x})A \wedge (\exists \mathbf{x})B \to (\exists \mathbf{x})(A \wedge B)$.

7. Let me ignore the content of Exercise 5 and try to prove
$$\vdash (\forall \mathbf{x})(\exists \mathbf{y})A \to (\exists \mathbf{y})(\forall \mathbf{x})A$$

Here it goes: By the DThm I prove as follows:

(1)	$(\forall \mathbf{x})(\exists \mathbf{y})A$	$\langle$hyp$\rangle$
(2)	$(\exists \mathbf{y})A$	$\langle$(1) + *spec*$\rangle$
(3)	$A[\mathbf{y} := \mathbf{z}]$	$\langle$aux. hyp for (2): **z** fresh$\rangle$
(4)	$(\forall \mathbf{x})A[\mathbf{y} := \mathbf{z}]$	$\langle$(3) + *gen*; OK: **x** not free in "Γ" (line (1))$\rangle$
(5)	$(\exists \mathbf{y})(\forall \mathbf{x})A$	$\langle$(4) + Dual *spec*"$\rangle$

Hmm … If you believe what *you* did in Exercise 5 above, then my "proof" above *must* be wrong, right?

Where *exactly* (which step) did I mis-step, and Why is the step wrong, *exactly*?

8. Use the $\exists$ elimination technique —and ping-pong if/where needed— to show

- $\vdash (\exists \mathbf{x})B \to (\exists \mathbf{x})(A \vee B)$.

  and

- $\vdash (\exists \mathbf{x})(A \wedge B) \equiv (\exists \mathbf{x})(A \vee B \equiv A \equiv B)$.

9. Let ψ be a binary predicate. Is $(\forall x)(\forall y)\psi(x, y) \to (\forall y)\psi(y, y)$ an instance of **Ax2**? **Why**?

10. Regardless of how you answered Problem 9, *prove* $\vdash (\forall x)(\forall y)\psi(x, y) \to (\forall y)\psi(y, y)$.

11. Prove "$\exists$-Monotonicity": If $\vdash A \to B$ then also $\vdash (\exists \mathbf{x})A \to (\exists \mathbf{x})B$.

12. Prove If $\vdash A \equiv B$ then also $\vdash (\exists \mathbf{x})A \equiv (\exists \mathbf{x})B$.

Caution! The long way (*does not get full marks*!) is to use ping-pong and Problem 11.