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MATH 1028Z. MID TERM TAKE-HOME (For ALL), March 4, 2024; 13:30-14:30
Professor George Tourlakis

Question 1. (4 MARKS) Prove that the relation $\subseteq$ —where **NO** *left/right* fields are restricting it— is a proper class.

Proof. For *every* SET A, we have $\emptyset \subseteq A$. But **ALL the sets A collected into a class** form the proper class $\mathbb{V}$.

Thus

$$\mathbb{V} \subseteq \text{ran}(\subseteq) \tag{1}$$

Since all sets ARE in $\mathbb{V}$, we also trivially have also $\text{ran}(\subseteq) \subseteq \mathbb{V}$ which turns (1) into

$$\text{ran}(\subseteq) = \mathbb{V} \tag{2}$$

We know however from class and notes (4.1.5) that if $\mathbb{P}$ is a relation that *IS* a set, then $\text{ran}(\mathbb{P})$ is a set. By (2) $\text{ran}(\subseteq)$ is a proper class. So $\subseteq$ CANNOT be a set. □

Question 2. (4 MARKS) Suppose $\mathbb{A} \subseteq \mathbb{B}$. Prove that if $\mathbb{A}$ is a proper class, then so is $\mathbb{B}$.

Proof. By contradiction: So **Let** $\mathbb{B}$ be a set. Then so is $\mathbb{A}$ by the "subclass theorem". This contradiction to the problem's main assumption implies that our red "Let" is false. So $\mathbb{B}$ is a proper class. □

Question 3. (4 MARKS) Prove that the **equality relation**, $=$, acting on **all objects of set theory**, that is, **on ALL sets and atoms**, **is a proper class**.

Proof. **Suppose instead** that $=$ is a *set relation*. Then by Theorem 4.1.5 —from class/web Notes— $\mathrm{dom}(=)$ **is a set as well.**

But $\mathrm{dom}(=)$ **contains all sets and atoms**, since $=$ is the class $\{(x, x)$: for *all* sets and atoms $x\}$.

That is, $\mathrm{dom}(=) = \mathbb{U}$, a **proper class**.

We just contradicted Theorem 4.1.5, so the **opposite** of what we assumed is true: $=$ is a non set class; a proper class. □

Question 4. (a) (4 MARKS) For any classes $\mathbb{A}, \mathbb{B}$ show that $\mathbb{A} \cap (\mathbb{A} \cup \mathbb{B}) = \mathbb{A}$.

(b) (4 MARKS) For any classes $\mathbb{A}, \mathbb{B}$ show that $\mathbb{A} \cup (\mathbb{A} \cap \mathbb{B}) = \mathbb{A}$.

Caution. In each case you must show that BOTH sides of "$=$" have the same elements. **RECOMMENDED** to use the technique "Assume $x \in lhs$. Here is my proof for $x \in rhs$". Repeat with the other direction: "Assume $x \in rhs$. ETC., ETC."

Proof.

(a′) For (a) I need to show

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$$\mathbb{A} \cap (\mathbb{A} \cup \mathbb{B}) \subseteq \mathbb{A} \qquad (a_1)$$

Here it is.

Let $x \in lhs$ of "$\subseteq$". By definition of "$\cap$", $x \in$ the leftmost "$\mathbb{A}$" in (a_1). Hence, trivially, is also in the rightmost "$\mathbb{A}$" (the rhs of "$\subseteq$") *DONE* (a_1).

and, also show

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$$\mathbb{A} \cap (\mathbb{A} \cup \mathbb{B}) \supseteq \mathbb{A} \qquad (a_2)$$

Here it is.

Let $x \in rhs$ of "$\supseteq$". By definition of "$\cup$", $x \in \mathbb{A} \cup \mathbb{B}$ as it needs to only be in one or the other operands of "$\cup$"; it is in $\mathbb{A}$. Now x being in both $\mathbb{A}$ and in $\mathbb{A} \cup \mathbb{B}$ it is in their *intersection*, i.e., in lhs of "$\supseteq$". *DONE* (a_2).

(b′) For (b) I need to show

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$$\mathbb{A} \cup (\mathbb{A} \cap \mathbb{B}) \subseteq \mathbb{A} \qquad (b_1)$$

Here it is.

Let $x \in lhs$ of "$\subseteq$". By definition of "$\cup$", I have two cases:

i. $x \in$ the leftmost "$\mathbb{A}$" in (b_1). Hence, trivially, is also in the rightmost "$\mathbb{A}$" (the rhs of "$\subseteq$") *DONE* (b_1).

ii. $x \in \mathbb{A} \cap \mathbb{B}$. Then $x \in \mathbb{A}$ by def. of $\cap$ and we are *DONE with the last case for* (b_1).

and, also show

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$$\mathbb{A} \cup (\mathbb{A} \cap \mathbb{B}) \supseteq \mathbb{A} \qquad (b_2)$$

Here it is.

Let $x \in rhs$ of "$\supseteq$". By definition of "$\cup$", $x \in \mathbb{A} \cup (\mathbb{A} \cap \mathbb{B})$ as it needs to only be in one or the other operands of "$\cup$". *DONE* (b_2).

□