Lassonde School of Engineering

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EECS 1028 Z. Problem Set No3 —SOLUTIONS

Posted: Mar. 22, 2024

1. (4 MARKS) Show that if $\mathbb{F}$ is a function and $\text{dom}(\mathbb{F})$ is a set then $\mathbb{F}$ is a set.

Proof. The function is $\mathbb{F} : \text{dom}(\mathbb{F}) \to \text{ran}(\mathbb{F})$.

In particular it is onto $\text{ran}(\mathbb{F})$ —every function $\mathbb{G} : \mathbb{A} \to \mathbb{B}$ is onto its range: Any $b \in \text{ran}(\mathbb{G})$ implies that b is one of the generated outputs, **meaning** $(\exists x \in \text{dom}(\mathbb{G}))\mathbb{G}(x) = b$.

Thus $\mathbb{F}[\text{dom}(\mathbb{F})] = \text{ran}(\mathbb{F})$ (Because: $\mathbb{F}[\text{dom}(\mathbb{F})] \subseteq \text{ran}(\mathbb{F})$ is trivial —all the outputs of a function go in its range; by definition of range. The "$\supseteq$" is by the argument above.

By a theorem in the Notes (**5.1.9**), $\text{ran}(\mathbb{F})$ is a set since $\text{dom}(\mathbb{F})$ is.

Hence so is $\mathbb{F} \subseteq \text{dom}(\mathbb{F}) \times \text{ran}(\mathbb{F})$ by the subclass theorem. □

2. (3 MARKS) True or False and WHY? (without the correct "WHY" this maxes out to 0 (zero) Marks). If $\mathbb{P}$ is a function and $\text{ran}(\mathbb{P})$ is a set, IS then $\mathbb{P}$ a set?

Answer. **False**. Consider the function below $\mathbb{P}$ (trivially single-valued: thus a function). It has $\text{ran}(\mathbb{P}) = \{0\}$, a set.

$$\mathbb{P} = \Big\{(x, 0) : x = x\Big\}$$

If this function is a set then so is its domain by a known theorem from class (**4.1.5**).

However $\text{dom}(\mathbb{P}) = \{x : x = x\} = \mathbb{U}$ that we know is a **proper** class. □

3. (3 MARKS) Prove that if the function f is 1-1, then f^{-1} —the converse of the relation f— is also a function.

Caution! The ONLY assumptions here are

1) f is a function and

2) it is 1-1.

f MAY be nontotal, non onto and have a lot of other "non" properties that you may HOWEVER NEITHER assume, NOR negate! Either way they are IRRELEVANT to the question!! **You MAY ONLY ASSUME WHAT I GAVE YOU HERE!!**

Proof. Given that f is 1-1, hence for all x, y, z we have

$$xfz \wedge yfz \to x = y \tag{1}$$

Let me write the above in terms of "f^{-1}", the converse RELATION.

$$zf^{-1}x \wedge zf^{-1}y \to x = y \tag{2}$$

(2) says that the RELATION "f^{-1}" is SINGLE-VALUED; A **FUNCTION**. □

4. Given a relation $R : A \to A$. Prove

(a) (2 MARKS) $\Delta_A \circ R = R$

Proof.

- **Do** $\Delta_A \circ R \subseteq R$: Let $x\Delta_A \circ Ry$. Then $x\Delta_A z$ **and** zRy for some z. But $x = z$ by def. of Δ_A. Thus the red part becomes xRy.
- **Do** $R \subseteq \Delta_A \circ R$: Let xRy. Then also

$$\underbrace{x\Delta_A x}_{Def.\ of\ \Delta_A} \quad Ry$$

Therefore $x\Delta_A \circ Ry$. □

(b) (2 MARKS) $R \circ \Delta_A = R$.

Proof.

- **Do** $R \circ \Delta_A \subseteq R$: Let $xR \circ \Delta_A y$. This says xRz **and** $z\Delta_A y$ for some z. By Δ-definition, $z = y$ and the red part becomes xRy. Done.

- **Do** $R \subseteq R \circ \Delta_A$: Let xRy. We also have $y\Delta_A y$ by Δ_A-definition. Thus $xRy\Delta_A y$, hence $xR \circ \Delta_A y$. Done. □

5. Let $f : A \to B$ be a 1-1 correspondence. **Then Prove**:

- (3 MARKS) $f^{-1} : B \to A$ is also a 1-1 correspondence.
  Proof.
  (a) f^{-1} is a function by Exercise 3. above.
  (b) f^{-1} is 1-1. Indeed, **Let** $xf^{-1}y$ and $zf^{-1}y$. This means the same as (definition of "f^{-1}") yfx and yfz. Since f is a function (single-valued) $x = z$. This conclusion and the red "**Let**" assumption establish that f^{-1} is 1-1.
  (c) **By assumption, f is total on A and onto B.** From Notes/Class (**4.4.15**) we have

$$A = \mathrm{dom}(f) = \mathrm{ran}(f^{-1}) \tag{1}$$

  and

$$\mathrm{dom}(f^{-1}) = \mathrm{ran}(f) = B \tag{2}$$

  (1) and (2) prove that f^{-1} is total on B and onto A. □

- (2 MARKS) If $gf = \mathbf{1}_A$, then we have $g = f^{-1}$ where f^{-1} is the converse of f.
  Proof. Note that our f is the same as above, a 1-1 correspondence $A \overset{f}{\sim} B$.
  Now apply f^{-1} to the right side of the given equality:

$$(gf)f^{-1} = \mathbf{1}_A f^{-1} \overset{exerc.\ 4}{=} f^{-1} \tag{3}$$

  On the other hand,

$$(gf)f^{-1} \overset{composition\ is\ assoc.}{=} g(ff^{-1}) = g\mathbf{1}_B \overset{4}{=} g \tag{4}$$

  We are done by (3) and (4).
  Wait! Why is $ff^{-1} = \mathbf{1}_B$? Because by Exercise 5c above, **for ANY** $x \in B$, it is $f^{-1}(x) = y$ for a unique $y \in A$. Thus, $f(y) = x$ by definition of converse.
  Substituting y by $f^{-1}(x)$ we obtain $ff^{-1}(x) = f\Big(f^{-1}(x)\Big) = f(y) = x \in B$. That is, is $ff^{-1} = \mathbf{1}_B$. □

- (2 MARKS) If $fh = \mathbf{1}_B$, then we have $h = f^{-1}$ where f^{-1} is the converse of f.

  Proof. Similar to the proof of the previous bullet:

  Note that our f is the same as above, a 1-1 correspondence $A \overset{f}{\sim} B$. Now apply f^{-1} to the LEFT side (this time) of the given equality:

  $$f^{-1}(fh) = f^{-1}\mathbf{1}_B \overset{exerc.\ 4}{=} f^{-1} \tag{5}$$

  On the other hand,

  $$f^{-1}(fh) \overset{composition\ is\ assoc.}{=} (f^{-1}f)h = \mathbf{1}_A h \overset{4}{=} h \tag{6}$$

  We are done by (5) and (6).

  Wait! Why is $f^{-1}f = \mathbf{1}_A$? Because by Exercise 5c above, **for ANY** $x \in A$, it is $f(x) = y$ for a unique $y \in B$. Thus, $f^{-1}(y) = x$ by definition of converse.

  Substituting y by $f(x)$ we obtain $f^{-1}f(x) = f^{-1}\Big(f(x)\Big) = f^{-1}(y) = x \in A$. That is, is $f^{-1}f = \mathbf{1}_A$.

  □

6. (4 MARKS) Let $<$ be an abstract (strict) order and $\mathbb{B}$ be any class.

Prove that $< \mid \mathbb{B}$ is an order **on** $\mathbb{B}$.

Hint. The notation "$< \mid B$" is given in the online Notes (where this Exercise is suggested for practice).

Proof. The notation $< \mid \mathbb{B}$ means $< \cap (\mathbb{B} \times \mathbb{B})$.

First off, for the "**on** $\mathbb{B}$" part, whatever kind of **relation** "$< \cap (\mathbb{B} \times \mathbb{B})$" proves to be it is a relation (a class of pairs) that is $\subseteq \mathbb{B} \times \mathbb{B}$. So the relation $< \cap (\mathbb{B} \times \mathbb{B})$ is "on $\mathbb{B}$".

So I prove that the latter is an order:

- **Irreflexive**: $(x, y) \notin <$, for all $x = y$ since $<$ is an order. But then such a pair (x, y) cannot be in the intersection $< \cap (\mathbb{B} \times \mathbb{B})$ either. This proves that $< \cap (\mathbb{B} \times \mathbb{B})$ is irreflexive.

- **Transitive**: Let (x, y) and (y, z) be in $< \cap (\mathbb{B} \times \mathbb{B})$.
  So,
  (a) the two pairs are in $\mathbb{B} \times \mathbb{B}$ in particular, and thus, all of x, y, z are in $\mathbb{B}$.
  (b) the two pairs are also in $<$ and since this is an order (hence transitive) we have $(x, z) \in <$.

  Since x, z are in $\mathbb{B}$ by item (a), we have $(x, z) \in (\mathbb{B} \times \mathbb{B})$. This and the previous sentence imply that $(x, z) \in < \cap (\mathbb{B} \times \mathbb{B})$. Done. □

7. Suppose we know that each of A_n, $n \geq 0$, is countable.

Then do the following:

(a) (3 MARKS) Prove that $\{A_i : i \in \mathbb{N}\}$ is a set.
If you used some of the Principles 0–3 in this subquestion, be explicit!
Hint. The countability of the A_n is irrelevant to this subquestion.
Proof. Each A_n has a unique label from $\mathbb{N}$. Since that qualifies the assignment of these labels/indices as a valid labelling (no two different sets among the A_i have the same label) and since the label class $\mathbb{N}$ is a set, then by Principle 3, the family $F = \{A_0, A_1, A_2, A_3, \ldots\}$ is a set. □

(b) (4 MARKS) Prove that $\bigcup\{A_i : i \in \mathbb{N}\} = \bigcup_{i \geq 0} A_i$ is countable.
Proof. Let A_n be enumerated as

$$A_n = \Big\{a_{n,0}, a_{n,1}, a_{n,2}, a_{n,3}, \ldots\Big\}$$

Arrange all these enumerations as rows in an infinite × infinite Matrix and traverse as shown by the NE arrows to effect an enumeration of $\bigcup\{A_i : i \in \mathbb{N}\}$.

$$\begin{array}{ccccccc}
a_{0,0} & & a_{0,1} & & a_{0,2} & & a_{0,3} \quad \ldots \\
 & \nearrow & & \nearrow & & \nearrow & \\
a_{1,0} & & a_{1,1} & & a_{1,2} & & \\
 & \nearrow & & \nearrow & & & \\
a_{2,0} & & a_{2,1} & & & & \\
 & \nearrow & & & & & \\
a_{3,0} & & & & & & \\
\vdots & & & & & &
\end{array}$$

□

(c) (2 MARKS) Did you need the Axiom of Choice in any of the two subquestions above?

Explain WHY clearly —in a FEW words— you had to, or did not have to.

Answer. The Axiom of Choice is technically needed in the above subquestion (b) only. **Each A_n has infinitely many enumerations** and I need to choose ONE row out of EACH ONE of these **infinitely many enumerations**. A mathematical "agent" that will do this for me is the Axiom of Choice.

> While we must be **aware** when the Axiom is **needed** (namely, when I am facing in my *PROOF* infinitely many choices that I *CANNOT* describe *FINITELY*), nevertheless in this **introductory course** we are content with just **awareness**. **We are not asked to, and we do not explicitly, show how exactly we use the Axiom.**

□

8. (a) (1 MARK) What does the name $\mathbb{V}$ stand for?

Answer. This is the proper class of all sets. It is $\mathbb{U}$ with all atoms removed. □

(b) (6 MARKS) Prove that the relation $\sim$ **on** $\mathbb{V}$ is *symmetric*, *transitive* and *reflexive*.

Proof.

Reflexive For any $A \in \mathbb{V}$, I have $A \sim A$.
The identity function $\mathbf{1}_A : A \to A$ is the 1-1 correspondence in this case.

Symmetric Let $A \sim B$ because $f : A \to B$ is a 1-1 correspondence. We saw (in Exercise 5) that $f^{-1} : B \to A$ is also a 1-1 correspondence. Thus $B \sim A$.

Transitive Let $f : A \to B$ and $g : B \to C$ be 1-1 correspondences. Then so is $gf : A \to C$, i.e., $A \overset{gf}{\sim} C$.
Indeed, $gf : A \to C$ is total, 1-1, and onto.

Total Is $gf(x)\downarrow$ for all $x \in A$? Well, $gf(x) = g\Big(f(x)\Big)$ and $f(x)\downarrow$. So $f(x)$ is an object in B. But $g(b)\downarrow$ for all objects in B. So $g(f(x))\downarrow$ for all $x \in A$.

Onto We want to show that the equation

$$gf(x) = c \tag{1}$$

has an x-solution for all $c \in C$. Well, $g(y) = c$ has solutions for all $c \in C$ since g is onto C.
We can now solve (1):

- First find $y \in B$ for $g(y) = c$. I can do that as g is onto.
- As f is onto B, I can find $x \in A$ so that $f(x) = y$.

  We have $g(f(x)) = g(y) = c$. We solved (1) —solution is x— since $fg(x) = f(g(x)) = c$.

1-1 Prove that gf or $f \circ g$ is 1-1. Assume, in relational notation that

$$xf \circ gz \wedge yf \circ gz \tag{2}$$

and prove $x = y$. First, (2) implies that $xfwgz$ for some w and $yfugz$ for some u.
Since g is 1-1, we have $w = u$. Then we have xfw and yfw.
Since f is 1-1, it is $x = y$. Done. □