Lassonde School of Engineering

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EECS 1028 Z. Problem Set No1 —SOLUTIONS

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1. True or False **and Why**. (**NOTE**: NO Why = NO Points)

(a) (2 MARKS) $\{\{a\},\{b\}\} = \{a, b\}$
FALSE. The "Why":

Case **1.** $a = b$. Then $\{a\} = \{b\}$ and the question becomes "$\{\{a\}\} = \{a\}$?" If yes, then $\{a\} = a$ hence $a \in a$. A *contradiction*.

Case **2.** $a \neq b$. If so, $\{a\} \neq \{b\}$ as well (equality *requires* $a = b$). We have two subcases since both sides have two elements:

A. $a = \{a\}$ and $b = \{b\}$. This is **FALSE**. For example, $a = \{a\}$ implies $a \in a$ that we know is impossible.

B. $a = \{b\}$ and $b = \{a\}$. This is *also* **FALSE**, else by substitution we have $a = \{\{a\}\}$. This implies $\{a\} \in a$, hence we have

$$a \text{ built before } \{a\} \text{ built before } a$$

A *contradiction*! □

(b) (2 MARKS) $\emptyset \in \emptyset$.

FALSE. Why? By definition of "$\emptyset$", $x \in \emptyset$ is FALSE for ALL x. In particular is false for $x = \emptyset$. □

(c) (2 MARKS) $\bigcup\{\{c\},\{d\}\} = \{c, d\}$

TRUE. By definition of $\bigcup$, lhs is what we get is the *set* built by "emptying" $\{c\}$ and $\{d\}$ inside an empty pair of braces $\{\quad\}$. But that is the rhs! □

(d) (2 MARKS) $\emptyset \subseteq \emptyset$

TRUE. We want, for any x, $\overbrace{x \in \emptyset}^{\mathbf{f}} \to x \in \emptyset$.
The labelling of the lhs of "$\to$" shows that the implication is true. □

(e) (2 MARKS) $\emptyset \in \{1\}$
FALSE. The only contents of rhs is "1" —an atom— which does not equal $\emptyset$ —a set. □

2. (3 MARKS) Is the class $\{\{x\} : \text{all } \underline{\text{atoms}}\ x\}$ a set? **Why** yes or no exactly?

Answer. YES!

The Why: All atoms are available at stage 0. Thus, *at stage 1* we can build each $\{x\}$ where x is an atom.

But then, *at stage 2* we can build the *class* containing *ALL* such $\{x\}$ as a *set*. □

3. (5 MARKS) Is the class $\mathbb{F} = \{\{x, y, z\} : \text{for all } \underline{\text{sets}} \text{ and } \underline{\text{atoms}}\ x, y, and\ z\}$ a set? **Why** yes or no exactly?

Answer. NO, it is a proper class. Why? Because IF $\mathbb{F}$ is a **SET, THEN**

1. $\{\{x\} : \text{for all } \underline{\text{sets}} \text{ and } \underline{\text{atoms}}\ x\}$ is ALSO a SET by the **subclass theorem** since $\{\{x\} : \text{for all } \underline{\text{sets}} \text{ and } \underline{\text{atoms}}\ x\} \subseteq \mathbb{F}$.
2. Hmm. **YET**, $\mathbb{U} = \bigcup\{\{x\} : \text{for all } \underline{\text{sets}} \text{ and } \underline{\text{atoms}}\ x\}$ (the lhs contains precisely ALL x without the "{ }" around them). So $\mathbb{U}$ is a set.† *Contradiction*!

□

4. (3 MARKS) Let A, B, C be sets or atoms. Prove that $\{A, B, C\}$ *is a set, without* using any of Principles 0, 1, 2. *Rather use results (theorems)* that we already established in class/Notes.

Proof. $\{A, B\}$ and $\{C\}$ (because it equals $\{C, C\}$) are sets (theorem for (not ordered) Pair). But then so is $\{A, B, C\} = \{A, B\} \cup \{C\}$ by union theorem. □

†Union theorem.

5. (5 MARKS) Prove that Principle 2 implies that we have infinitely many stages available.

Hint. Arguing by contradiction, assume instead that we only have **finitely many** stages. So repeatedly applying Principle 2 we can form a non ending sequence of stage names

$$\cdots < \Sigma' < \Sigma'' < \Sigma''' < \Sigma'''' < \cdots \tag{1}$$

If the sequence (1) contains only a *finite* number of distinct $\Sigma^{''\cdots'}$, then at least two of the $\Sigma^{''\cdots'}$ in (1) are the same stage. Use this conclusion and properties of "<" to get a contradiction.

Proof.

> We are using Principle 2 as: "given stage Σ. Then there is a stage Σ' after it, that is, $\Sigma < \Sigma'$."

Assuming only finitely many stages, the stages themselves named, in (1) above, at some point *repeat*, that is,

> two names Σ_i and Σ_j in the sequence (1) name the same stage. We can say this as $\Sigma_i = \Sigma_j$.

So, we have the situation below, where I am switching to subscript notation it being more user friendly than "accent" notation

$$\cdots \Sigma_i < \Sigma_{i+1} < \cdots < \Sigma_{j-1} < \Sigma_j \cdots$$

By transitivity of "<", we have $\Sigma_i < \Sigma_j$ which is impossible since the two stage names Σ_i and Σ_j name the same stage. □

6. (4 MARKS) Prove that, for any *set* A we have that $\mathbb{U} - B$ is *a proper class*.

Proof. See also the posted "news" item with date Jan. 29.

So by notation, B is a set. **The set A is irrelevant to the question as it does not relate to the conclusion. We ignore it.**

We argue that $\mathbb{U} - B$ is a *proper class* by contradiction.

So **assume otherwise, that $\mathbb{U} - B$ is a set.**

By the union theorem so is $(\mathbb{U} - B) \cup B$.

But the above union equals $\mathbb{U}$ and *we have a contradiction* as this implies that $\mathbb{U}$ is a set.

To believe the above claim of equality we note that $(\mathbb{U} - B) \cup B \subseteq \mathbb{U}$ since $\mathbb{U}$ contains every set and atom.

For $\mathbb{U} \subseteq (\mathbb{U} - B) \cup B$ let $x \in lhs$ (of "$\subseteq$"). We have two cases:

Case **1.** $x \in B$. Then $x \in rhs$ by definition of Union.

Case **2.** $x \notin B$. Since $x \in \mathbb{U}$ then $x \in \mathbb{U} - B$ by def. of "$-$". Then $x \in rhs$ by definition of Union.

$\square$

7. (4 MARKS) Prove for any classes $\mathbb{A}, \mathbb{B}$, that $\mathbb{A} - \mathbb{B} = \mathbb{A} - \mathbb{A} \cap \mathbb{B}$.

Hint. **This is a simple case of proving** $lhs \subseteq rhs$ **by doing "Let** $x \in lhs$**. BLA BLA BLA and concluding** $x \in rhs$**", and then** ***ALSO*** **doing** $rhs \subseteq lhs$ **by doing "Let** $x \in rhs$**. BLA BLA BLA and concluding** $x \in lhs$**".**

Proof. **Please DO follow the Hint and NEVER MIND "de Morgan Law" and other "exotica" that we have not covered —which means, if you use it, you must prove it!!**

Two directions:

$\subseteq$ Case. Let $x \in lhs$. Then

$$x \in \mathbb{A} \tag{1}$$

and

$$x \notin \mathbb{B} \tag{2}$$

By (2), we have $x \notin \mathbb{A} \cap \mathbb{B}$ (the opposite requires $x \in \mathbb{B}$). This and (1) mean (by def of "$-$") $x \in rhs$.

$\supseteq$ Case. Let $x \in rhs$. Then

$$x \in \mathbb{A} \tag{3}$$

and

$$x \notin \mathbb{A} \cap \mathbb{B} \tag{4}$$

By (3, 4), we *CANNOT* have $x \in \mathbb{B}$ (else along with (3) we contradict (4)). So it is

$$x \notin \mathbb{B} \tag{5}$$

(3) and (5) jointly prove $x \in lhs$. □

8. Use notation by explicitly listing **all the members** of each rhs $\{???\}$ to complete the following incomplete equalities:

This is a "handout"! We have done it in class!

Answers.

(a) (2 MARKS) $2^{\emptyset} = \{\emptyset\}$

(b) (2 MARKS) $2^{\{1,2,3\}} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

□