Lassonde School of Engineering

Dept. of EECS
Professor G. Tourlakis
EECS 1028 Z. Practice Problem Set —Prep. for Exam; NOT for Submission or Credit
Posted: March 26, 2024

1. Prove that $\mathbb{U} \times \mathbb{U}$ is a proper class.

Proof. This class is a class of pairs: **A Relation**.

Suppose this Relation is a *SET*.

Then via a theorem from Notes/Class, $\text{dom}(\mathbb{U} \times \mathbb{U}) = \mathbb{U}$ is a SET. But we know that $\mathbb{U}$ is a proper class, which goes against assumption that starts with "**Suppose**". Done. □

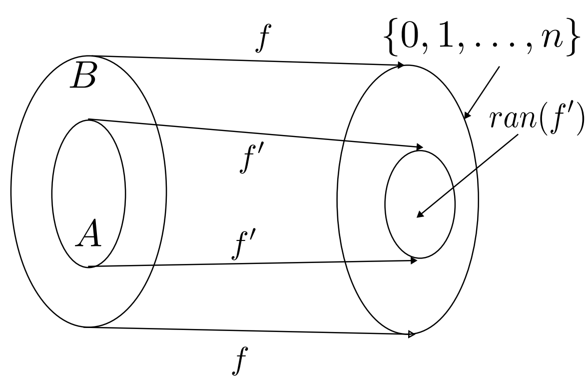
2. Prove that if B is finite and $A \subseteq B$, then A is also finite.

Proof.

(a) Case where $B = \emptyset$. Then $A = \emptyset$ since this is the only subset of $\emptyset$. By definition then, A is finite.

(b) Case where $B \overset{f}{\sim} \{0, 1, 2, \ldots, n\}$, that is

$$f : B \to \{0, \ldots, n\} \text{ is a 1-1 correspondence}$$



Define the restriction of f on A —we will call it f'— for all $x \in A$, by

$$f'(x) = f(x) \tag{1}$$

f' is NOT defined on the rest of B, namely on $B - A$.

Since f is total on B and thus defined everywhere in A, f' is total on A. f' is also 1-1 since $xf'y \wedge zf'y$ implies $xfy \wedge zfy$ by (1). Now f being 1-1 implies $x = z$ and thus f' is 1-1.

We have many times said (and shown why) that **every function is onto its range**. Thus $f' : A \to \text{ran}(f')$ is a 1-1 correspondence

$$A \sim \text{ran}(f') \tag{2}$$

Next we note that $\text{ran}(f')$ is finite: Indeed, **arguing by contradiction**, if $\text{ran}(f')$ is *infinite*, then —from $\text{ran}(f') \subseteq \{0, 1, \ldots, n\} \subseteq \mathbb{N}$, and a theorem from Notes/class,

$$\text{ran}(f') \sim \mathbb{N} \tag{3}$$

Let $g : \text{ran}(f') \to \mathbb{N}$ effect this 1-1 correspondence. The function $h : B \to \mathbb{N}$ given by

$$h(x) = \begin{cases} g(x) & \text{if } x \in \text{ran}(f') \\ \uparrow & \text{if } x \in B - \text{ran}(f') \end{cases}$$

is onto $\mathbb{N}$ contradicting a theorem from class/Notes.

We conclude that $\text{ran}(f') \sim \{0, 1, \ldots, r\}$ for some r —i.e., is finite— and combining with (2) we have $A \sim \text{ran}(f') \sim \{0, 1, \ldots, r\}$. Hence $A \sim \{0, 1, \ldots, r\}$ by $\sim$-transitivity. A is finite! Done. □

3. Prove that an enumerable set is infinite.

Proof. Let $A \sim \mathbb{N}$ which is saying "A is enumerable" mathematically.

If A is also finite then $\{0, 1, \ldots, m\} \sim A$ for some $m \in \mathbb{N}$. Thus

$$\{0, 1, \ldots, m\} \sim A \sim \mathbb{N}$$

Hence (transitivity of $\sim$) we have

$$\{0, 1, \ldots, m\} \sim \mathbb{N}$$

which implies an **onto function** (the 1-1 correspondence "$\sim$")

$$\{0, 1, \ldots, m\} \to \mathbb{N}$$

But our Notes say that this is impossible! □

4. Let A be *enumerable*. Show how —given an enumeration of A **without repetitions**— you can construct a **NEW** enumeration where **EACH** $x \in A$ **is enumerated infinitely many times**.

Proof. Let

$$a_0, a_1, a_2, \ldots, a_i, \ldots \tag{1}$$

be an enumeration without repetitions.

Form the infinite "square" matrix whose every row is the same as the sequence (1). Then traverse it with SE arrows as in the display below.

$$\begin{array}{cccccccc} a_0 & & a_1 & & a_2 & & a_3 & \ldots \\ & \nearrow & & \nearrow & & \nearrow & & \\ a_0 & & a_1 & & a_2 & & & \\ & \nearrow & & \nearrow & & & & \\ a_0 & & a_1 & & & & & \\ & \nearrow & & & & & & \\ a_0 & & & & & & & \\ \vdots & & & & & & & \end{array}$$

The above matrix includes each a_i infinitely many times (for each n, column n has all its entries equal to a_n). All these will be enumerated in the SE enumeration depicted above. Done. □

5. Prove that $\vdash (\forall x)A \to (\exists x)A$.

Proof. By DThm, prove instead

$$\vdash (\forall x)A \vdash (\exists x)A$$

1) $(\forall x)A$ ⟨hyp from DThm⟩
2) A ⟨1 + Spec⟩
3) $(\exists x)A$ ⟨2 + Dual Spec⟩

□

6. Prove that $\vdash (\forall x)(A \to B) \to (\exists x)A \to (\exists x)B$.

Proof. By DThm, prove instead

$$(\forall x)(A \to B) \vdash (\exists x)A \to (\exists x)B$$

and once more by DThm do instead:

$$(\forall x)(A \to B), (\exists x)A \vdash (\exists x)B$$

Here it goes:

1) $(\forall x)(A \to B)$ ⟨hyp from DThm⟩
2) $(\exists x)A$ ⟨hyp from DThm⟩
3) $A[c]$ ⟨Aux. Hyp for 2; c is not in concl. nor in 1 +2⟩
4) $A[c] \to B[c]$ ⟨1 + Spec⟩
5) $B[c]$ ⟨3 + 4 + MP⟩
6) $(\exists x)B[x]$ ⟨5 + Dual Spec⟩

□

7. Use simple induction to prove that $n + 10 < 3^n$, for $n \geq 3$.

Proof.

Basis. $n = 3$. To verify $3 + 10 < 27$. True!

I.H. Fix $n \geq 3$ and Assume

$$n + 10 < 3^n$$

for that n.

I.S. Prove

$$n + 1 + 10 < 3^{n+1}$$

Here:

$$\begin{aligned} n + 11 &< 3n + 30, \text{ recall, } n \geq 3 \\ &< \Big(3^n\Big)3, \text{ multiplying both sides of I.H. by 3} \\ &= 3^{n+1} \end{aligned}$$

□

8. Consider the statement (formula)

$$(\exists x)A(x) \rightarrow A(c) \tag{1}$$

where c is a *new* constant, NOT found in $A(x)$.

Find now a *specific* **SIMPLE** example of $A(x)$ over the set $\mathbb{N}$ and choose a specific value of $c \in \mathbb{N}$ so that (1) becomes **false**, and **Therefore** we **cannot** prove (1), since proofs start from true axioms and preserve truth at every step.

Proof. Since c is **NOT specified** by "$(\exists x)A(x)$" in any shape or form, **I am free to take** the special case below (over our familiar $\mathbb{N}$) and I choose, unimaginatively (:-), the constant "c" to be 42. I choose for "$A(x)$" the formula $x = 0$.

So statement (1) becomes

$$\overbrace{(\exists x)x = 0}^{\mathbf{t}} \rightarrow \overbrace{42 = 0}^{\mathbf{f}} \tag{2}$$

The rhs of $\rightarrow$ in (2) is **false**. Hence makes the whole simplified formula false. So (1) cannot be an always true formula of Logic! □

9. Define the closure $\mathrm{Cl}(\mathcal{I}, \mathcal{O})$ by the specifications

(a) $\mathcal{I} = \{2\}$

(b) The ONLY operation in $\mathcal{O}$ is

$$(x, y) \mapsto x + y \tag{1}$$

That is, if the operation gets input x and y it produces output $x + y$.

Prove by induction on $\mathrm{Cl}(\mathcal{I}, \mathcal{O})$ that all its members are even natural numbers.

Proof. The property of members x of $\mathrm{Cl}(\mathcal{I}, \mathcal{O})$ that I am asked to prove is "x is even".

Basis. Verify for members of $\mathcal{I}$. There is ONLY ONE member in this set, namely, the number 2. This IS EVEN!

Prove that the property propagates by the only rule, (1):

So, say the inputs x and y of (1) are even, $2n$ and $2m$ respectively.

Then so is the output of rule (1), because $x + y = 2n + 2m = 2(n + m)$. DONE! □

10. Using Simple Induction (SI) prove that $1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, for $n \geq 1$.

Proof.

Basis. $n = 1$. Verify: $lhs = 1^3 = 1$. $rhs = \Big((1 \times 2)/2\Big)^2 = 1^2 = 1$. Good!

I.H. Fix $n \geq 1$ and *Assume*

$$1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{1}$$

I.S. Prove for the n we fixed in the I.H. that

$$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2 \tag{2}$$

Here it goes:

$$\begin{aligned}
1^3 + 2^3 + \ldots + n^3 + (n+1)^3 &\overset{I.H.}{=} \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\
&= (n+1)^2 \left[\frac{n}{2}\right]^2 + (n+1)^3 \\
&= (n+1)^2 \left[n^2/4 + (n+1)\right] \\
&= (n+1)^2 \left[\frac{n^2 + 4n + 4}{4}\right] \\
&= (n+1)^2 \left[\frac{(n+2)^2}{4}\right] \\
&= \left[\frac{(n+1)(n+2)}{2}\right]^2
\end{aligned}$$

□