Consider the following program.

```
fun introot 0 = 0
  | introot n = increase(2 * introot(n div 4), n)

fun increase (k, n) =
  if (k + 1) * (k + 1) > n then k else k + 1
```

PROPOSITION *For all* n, *if* `introot` $n = k$ *then* $k^2 \leq n < (k+1)^2$.
PROOF We prove the above proposition by complete induction on n. We distinguish two cases.

- If $n = 0$, the above is vacuously true.
- Let $n > 0$. By induction, if `introot` $(n$ `div` $4) = k'$ then $k'^2 \leq n < (k'+1)^2$. In this case, we distinguish two situations.
  - If $(2k'+1)^2 > n$ then $k = 2k'$ and

$$\begin{aligned} k^2 &= (2k')^2 \\ &= 4k'^2 \\ &\leq 4(n \div 4) \\ &\leq n \\ &< (2k'+1)^2 \\ &= (k+1)^2. \end{aligned}$$

  - If $(2k'+1)^2 \leq n$ then $k = 2k'+1$ and

$$\begin{aligned} k^2 &= 2k'+1)^2 \\ &\leq n \\ &\leq 4(n \div 4) + 3 \\ &< 4(k'+1)^2 + 3 \\ &= 4k'^2 + 8k' + 4 \\ &= (2 * k' + 2)^2 \\ &= (k+1)^2. \end{aligned}$$

□