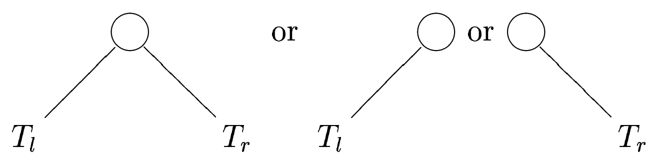
Proposition 7.2 *Let T be a nonempty AVL tree with n nodes and height h.*

1. $n > 2^{\frac{h}{2}-1}$
2. $h < 2\log(n) + 2$

Proof

1. We prove this proposition by structural induction on T. The base case, where T consists of a single node, is trivial ($h = 0$ and $n = 1$). Next we consider the induction step. The AVL tree T is either of the form



We only consider the first case. The other two cases can be handled similarly. Since the subtrees T_l and T_r are smaller than T, by the induction hypothesis the claim holds for these subtrees. Assume that the subtree T_l (T_r) has n_l (n_r) nodes and height h_l (h_r). Because T has the height-balance property, we have that

- T_l and T_r both have height $h-1$, or
- T_l has height $h-1$ and T_r has height $h-2$, or
- T_l has height $h-2$ and T_r has height $h-1$.

We only consider the last case. The others can be proved similarly.

$$\begin{aligned} n &= n_l + n_r + 1 \\ &> 2^{\frac{h_l}{2}-1} + 2^{\frac{h_r}{2}-1} + 1 \quad \text{[induction]} \\ &= 2^{\frac{h-2}{2}-1} + 2^{\frac{h-1}{2}-1} + 1 \quad [h_l = h-2 \text{ and } h_r = h-1] \\ &> 2 \cdot 2^{\frac{h-2}{2}-1} \\ &= 2^{\frac{h-2}{2}} \\ &= 2^{\frac{h}{2}-1}. \end{aligned}$$

2. According to 1.,

$$\begin{aligned} & n > 2^{\frac{h}{2}-1} \\ & \Rightarrow \log(n) > \frac{h}{2} - 1 \\ & \Rightarrow h < 2\log(n) + 2. \end{aligned}$$

□