PROPOSITION 6.5. *Let $T$ be a nonempty binary tree with $n$ nodes and height $h$.*

1. *If the levels 0, 1, ... , $h$ have the maximum number of nodes then $n = 2^{h+1} - 1$.*
2. *If the levels 0, 1, ... , $h - 1$ have the maximum number of nodes and level $h$ has one node*[1] *then $n = 2^h$.*
3. *If $T$ is complete then $2^h \leq n \leq 2^{h+1} - 1$.*
4. *If $T$ is complete then $h = \lfloor \log(n) \rfloor$.*

PROOF

1. Induction on $T$ (easy exercise).
2. Immediate consequence of 1.
3. Immediate consequence of 1. and 2.
4. According to 3.,

$$\begin{aligned} &2^h \leq n \\ &\quad \Rightarrow \quad h \leq \log(n). \end{aligned} \tag{1}$$

According to 3.,

$$\begin{aligned} &2^{h+1} - 1 \geq n \\ &\quad \Rightarrow \quad 2^{h+1} \geq n + 1 \\ &\quad \Rightarrow \quad h + 1 \geq \log(n + 1) \\ &\quad \Rightarrow \quad h \geq \log(n + 1) - 1 \\ &\quad \Rightarrow \quad h > \log(n) - 1 \quad [\log(n + 1) > \log(n)]. \end{aligned} \tag{2}$$

Combining (1) and (2) we get $h = \lfloor \log(n) \rfloor$.

$\square$

[1]This is not a strong binary tree.