PROPOSITION 6.5. *Let T be a nonempty binary tree with n nodes and height h.*

1. *If the levels 0, 1, ... , h have the maximum number of nodes then $n = 2^{h+1} - 1$.*
2. *If the levels 0, 1, ... , $h-1$ have the maximum number of nodes and level h has one node*[1] *then $n = 2^h$.*
3. *If T is complete then $2^h \leq n \leq 2^{h+1} - 1$.*
4. *If T is complete then $h = \lfloor \log(n) \rfloor$.*

PROOF

1. Induction on T (easy exercise).
2. Immediate consequence of 1.
3. Immediate consequence of 1. and 2.
4. According to 3.,

$$
\begin{aligned}
& 2^h \leq n \\
& \quad \Rightarrow \quad h \leq \log(n). \qquad (1)
\end{aligned}
$$

According to 3.,

$$
\begin{aligned}
& 2^{h+1} - 1 \geq n \\
& \quad \Rightarrow \quad 2^{h+1} \geq n + 1 \\
& \quad \Rightarrow \quad h + 1 \geq \log(n+1) \\
& \quad \Rightarrow \quad h \geq \log(n+1) - 1 \\
& \quad \Rightarrow \quad h > \log(n) - 1 \quad [\log(n+1) > \log(n)]. \qquad (2)
\end{aligned}
$$

Combining (1) and (2) we get $h = \lfloor \log(n) \rfloor$.

□

[1] This is not a strong binary tree.