PROPOSITION 6.5. *Let T be a nonempty binary tree with n nodes and height h.*

1. *If the levels 0, 1, ... , h have the maximum number of nodes then $n = 2^{h+1} - 1$.*
2. *If the levels 0, 1, ... , $h-1$ have the maximum number of nodes and level h has one node*[1] *then $n = 2^h$.*
3. *If T is complete then $2^h \leq n \leq 2^{h+1} - 1$.*
4. *If T is complete then $h = \lceil \log(n) \rceil$.*

PROOF

1. Induction on h (easy exercise).
2. Immediate consequence of 1.
3. Immediate consequence of 1. and 2.
4. According to 3., $2^h \leq n$. Hence,

$$h \leq \log(n). \tag{1}$$

According to 3., $2^{h+1} - 1 \geq n$. Therefore, $2^{h+1} \geq n + 1$ and $h + 1 \geq \log(n + 1)$. Hence, $h \geq \log(n + 1) - 1$, so

$$h \geq \log(n) - 1 \tag{2}$$

Combining (1) and (2) we get $h = \lfloor \log(n) \rfloor$.

□

[1] This is not a proper binary tree acccording to the textbook.