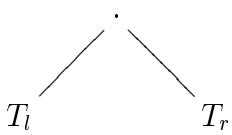
PROPOSITION 5.10.1 *Let T be a nonempty binary tree with e external nodes and height h. Then $h + 1 \leq e \leq 2^h$.*

PROOF We prove this claim by induction on the height h. In the base case ($h = 0$), the tree consists of a single external node and hence the claim is true. Next we consider the induction step. Let $h > 0$. Assume that the claim is true for all binary trees with height less than h. The binary tree T is of the form



Since the height of the subtrees T_l and T_r is less than h, by the induction hypothesis the claim holds for these subtrees. Assume that the subtree T_l (T_r) has e_l (e_r) external nodes and height h_l (h_r). Then

$$\begin{aligned}
h + 1 &\leq h_l + 1 + h_r + 1 \quad [h = \max\{h_l, h_r\} + 1] \\
&\leq e_l + e_r \quad [\text{induction hypothesis}] \\
&= e \\
&= e_l + e_r \\
&\leq 2^{h_l} + 2^{h_r} \quad [\text{induction hypothesis}] \\
&\leq 2 \cdot 2^{\max\{h_l, h_r\}} \\
&= 2^h \quad [h = \max\{h_l, h_r\} + 1]
\end{aligned}$$

□