FORMAL VERIFICATION OF A CONCURRENT BINARY SEARCH TREE

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Abstract

In this thesis, we formally verify a simplified version of the non-blocking linearizable binary search tree of Ellen et al., which appeared in the *Proceedings of the 29th Annual ACM Symposium on Principles of Distributed Computing* (pages 131-140), using the PVS specification and verification system. The algorithm and its specification are both modelled as I/O automata. In order to formally verify that the algorithm implements the specification, we show that the algorithm's I/O automaton simulates the specification's. An intermediate I/O automaton is constructed to simplify the simulation proof of linearizability. By showing there is a forward simulation from the algorithm's I/O automaton to the intermediate automaton and there is a backward simulation from the intermediate automaton to the specification's automaton, we formally verify that the algorithm implements its specification. While formalizing the proof, we found small errors in the original proof.

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1 Introduction

With the arrival of the multi-core central processing unit (CPU) revolution, a considerable fraction of the applications developed today is concurrent. “But concurrency is hard. Not only are today’s languages and tools inadequate to transform applications into parallel programs, but also it is difficult to find parallelism in main stream applications, and—worst of all—concurrency requires programmers to think in a way humans find difficult”, as Microsoft’s Herb Sutter and James Larus wrote in 2005 [1]. Unfortunately, almost a decade later, the above quote still reflects the current state of affairs in the field of software development.

Since concurrency is hard, libraries with concurrency primitives such as a concurrent array, a concurrent queue and a concurrent set are essential for today’s software developer. Such concurrency primitives provide objects that can be accessed concurrently by multiple processes. One popular way of specifying correctness of such a concurrent object is *linearizability*. This means that it behaves as if operations on it occur instantaneously. For example, to design a concurrent algorithm that stores a searchable set that can be updated by insert and delete operations, which is a very common scenario, a concurrent implementation of binary search trees will be very helpful.

It has been a long time since researchers realized that traditional approaches, which are based on mutual exclusion, are often not efficient enough and have associated problems such as deadlock, livelock and convoying. To combat these problems, they developed concurrent objects that do not rely on mutual exclusion mechanisms such as locks and semaphores. *Lock-free* implementations, which avoid mutual exclusion using other techniques to coordinate access by different processes to shared data, provide high parallelism and good performance under a variety of different workloads. There are several different progress properties that a lock-free algorithm can satisfy. The *wait-free* property [2] ensures that any process can complete any operation in a finite number of steps, regardless of the execution speeds of the other processes. The *non-blocking* property guarantees that some operation will complete in a finite number of steps. This weaker condition allows individual processes to starve but guarantees system-wide progress. All wait-free algorithms are non-blocking. The terms non-blocking and lock-free are often used interchangeably. We use non-blocking as a term to specify algorithms that satisfy this progress condition and use lock-free as a general term to describe algorithms that do not rely on mutual exclusion. Lock-free algorithms are generally more complex than lock-based algorithms, because intricate parallelism techniques are introduced to avoid locks.

Recently, Ellen et al. [3] gave a non-blocking linearizable implementation of a binary search tree. They provided a proof of correctness written in English which was quite lengthy and complex. The goal of our work is to provide a formal verification of this proof.

1.1 Formal Verification

Since concurrency is hard, it should come as no surprise that concurrent algorithms are prone to errors. Therefore, there are significant challenges to ensure the correctness of concurrent algorithms in general, and lock-free algorithms in particular. For example, Detlefs et al. [4] developed a lock-free double-ended queue, called the Snark algorithm, with a proof of correctness, but bugs were reported later by Doherty [5]. Shann et al. [6] published their non-blocking queue algorithm with safety proofs, but bugs were found when formal verification methods were applied by Colvin and Groves in [7].

Although it is hard to avoid bugs in an algorithm, there are a variety of methods to reduce the chance of making a mistake. Testing is one such method to detect bugs. However, testing all executions of a nontrivial concurrent algorithm is usually infeasible (or even impossible) because there are simply too many. Formal verification is another method of checking whether a design satisfies some properties. It is the act of proving or disproving the correctness of algorithms with respect to a certain formal specification or property, using formal methods of mathematics. The two main approaches are model checking and theorem proving.

Model checking builds a mathematical model of the algorithm as a collection of states and actions that move from one state to another. Then, a model checker performs a systematic, exhaustive exploration of the state space, for example to check that an invariant is true in all reachable states. Lamport [8] described a way to verify concurrent algorithms using the +CAL algorithm language and the TCL model checker and successfully detected the bugs in the Snark algorithm found by Doherty et al. [9]. He translated the algorithm into the +CAL algorithm language and then to a TLA+ specification that can be model checked. Liu et al. [10] presented their way to check linearizability based on refinement relations from abstract specifications to concrete implementations using model checking methods. Their method exploits model checking of finite state systems specified as concurrent processes with shared variables, and partial order reduction is applied to reduce the search space. The toolset they used can automatically check a variety of algorithms. However, no tree implementation has been verified in their paper and the tool has not been published. The approaches that use model checking techniques can help us quickly discover bugs in some algorithms. However, for complicated concurrent algorithms, the search spaces are usually too large to be explored in a reasonable amount of time and using a reasonable amount of space.

Theorem proving (also called deductive reasoning), is the proving of mathematical theorems in a format so that the proof can be checked by a computer program such as PVS, Coq, HOL or Isabelle. Theorem proving is more powerful than model checking because it can deal with an infinite state space more easily. Because the binary search tree of Ellen et al. [3] is a data structure of unbounded keys, and processes, the state space is infinite. So, we use the PVS theorem prover to formalize the binary search tree's proof of correctness.

PVS is a verification system that contains a specification language and a theorem prover. We used this verification system in our proofs. Compared with other theorem proving systems, such as HOL and Isabelle, PVS has several advantages. First, its specification language allows the user to define things in a way that is similar to programming languages. For instance, we can define a data type using square brackets and specify its fields' names and types. We can also define data types by explicitly writing *datatype*, and PVS then automatically generates basic axioms for it. Hence, having such a specification language, we can easily formalize pseudocode using PVS and focus on proving correctness. Second, unlike HOL, the PVS theorem prover allows users to define their axioms, which provides more freedom to construct proofs based on some facts without getting stuck on low level details. However, because introducing user defined axioms may introduce inconsistencies, axioms must be very carefully designed. Third, PVS also provides some automatic reasoning procedures such as *grind, assert* and *bddsimp* etc., which simplify the task of proving correctness.

1.2 Previous Work

Lynch and Tuttle [11] introduced I/O automata (IOA), which can be used to model concurrent algorithms and specifications of correctness. To prove that an algorithm correctly implements a specification, one has to show that for every execution of the algorithm's automaton, the externally observable behaviour is the same as the behaviour of the specification automaton. Lynch and Vaandrager [12] introduced techniques for doing this, including forward simulations and backward simulations.

In Doherty's Master's thesis [5], he introduced a way to use the PVS theorem prover to check the forward simulation between the Snark algorithm [4] and its specification, and detected bugs in the algorithm. A canonical automaton was introduced there to model the specification and capture the property of linearizability. Then, a forward simulation was used to show that the automaton of the concurrent implementation simulates the canonical automaton, thus showing that the implementation is linearizable. (We shall discuss this technique in more detail in Chapter 2.) While the author was trying to prove the correctness of the Snark algorithm using a forward simulation, he detected bugs. He also showed how to fix those bugs, and mentioned that a backward simulation might be needed to complete the proof. The complete verification of this algorithm was done later in [9].

Doherty et al. [13] used a similar method to verify a queue that is slightly optimized from Michael and Scott's non-blocking FIFO queue [14] using the PVS theorem prover. In this paper, they formally described a way to verify the non-blocking algorithm using simulation techniques. Their idea was to introduce three IOAs: an abstract one which was the same as the canonical automaton, an intermediate one and a concrete one which represented the implementation. A backward simulation was proved between the abstract IOA and the intermediate IOA, and a forward simulation was verified between the intermediate IOA and the concrete IOA. This technique was also used in [9] to complete the proof of the correctness of the modified Snark algorithm. More details about the relationship between backward simulations and non-blocking algorithms are also discussed in [15].

Colvin et al. [7] used this method (*i.e.*, using three IOAs) to prove the correctness of an array-based non-blocking implementation of a bounded queue by Shann et al. [6] and also detected errors in the algorithm. After they fixed that algorithm, they successfully formally verified the modified version using PVS.

In later work of Colvin et al. [16], they used this hybrid forward and backward simulation technique to verify a lazy concurrent list-based set algorithm [17]. Although this algorithm is based on locks, one of its operations (a *contain* operation) has the wait-freedom property.

Although many concurrent algorithms manipulating arrays, queues and lists have been formally verified, no one has tried to formally verify non-blocking binary search tree algorithms so far. Hence, our goal is to formally verify the non-blocking binary search tree algorithm discovered by Ellen et al. [3]. Our idea of how to formally verify the algorithm was inspired by Colvin et al. [16].

Many people have formally verified concurrent algorithms using the PVS theorem prover. Gao used PVS to prove the correctness and progress properties of a lock-free dynamic hash table in his Ph.D. thesis [18]. In his thesis, he also proved the correctness of a lock-free parallel garbage collector using PVS. Archer et al. [19] described a general way to model concurrent algorithms as timed I/O automata using the Tempo toolkits [20] and then verify the properties of the I/O automata in PVS through the interface TAME [21, 22].

1.3 Overview of the Thesis

In this thesis, we first introduce our model of computation in Chapter 2. For a data type, we define its sequential executions, atomic executions and concurrent executions. Once, an implementation of a data type is modelled by means of an I/O automaton, simulations are used to prove the correctness of its executions.

Chapter 3 presents the non-blocking binary search tree algorithm of Ellen et al. [3]. We give an overview of how this algorithm works and then explain some key steps of that algorithm. A simplified algorithm is given later in this chapter to make our proofs easier.

In Chapter 4, the details of modelling the binary search tree algorithm are presented. Then, we define a forward simulation and a backward simulation in order to prove the correctness of the algorithm. Chapter 5 discusses the proofs for the forward and backward simulation. The difficulties and bugs that we detected during the formal verification using PVS are also discussed there. Chapter 6 gives a summary and describes some of the future work we would like to pursue.

2 Proving Linearizability Using Simulations

2.1 Model of Computation

To model data types and algorithms in a shared-memory architecture, we use an *asynchronous shared-memory model*. In the asynchronous model, different processes take steps in an arbitrary order, at arbitrary relative speeds. Intuitively, in the asynchronous model, we assume there is a scheduler that determines which process will take the next step. Algorithms and data structures should behave correctly for all possible schedules made by that scheduler. Our model allows *failures* to model the fact that the systems in which the algorithms are running may not be completely reliable. Therefore, programs need to tolerate faulty behaviour. We consider *crash failures*: a process executing some code may stop without a warning. (These are also known as halting failures.)

In a *shared-memory model*, a collection of processes interact with one another via a collection of shared objects [23]. Such an assumption captures the way communication occurs in a multi-core CPU. We assume our system provides some *atomic* *shared objects*, either implemented in hardware or by the operating system. Atomic objects can be accessed concurrently by several processes, but they ensure that each operation performed by the processes occurs atomically. For example, *read/write registers* are one of the most frequently used types of shared objects in concurrent systems. A read/write register stores a value. A $write(v)$ operation changes the value to v and returns *ack*. A *read* operation returns the value currently stored in the read/write register without changing it. Both write and read operations are atomic. *Compare-and-swap* is also a popular type of atomic object in multi-core systems. A *compare-and-swap* (CAS) object X stores a value from a universe U, and provides only one operation, $CAS(u, v)$, where u and v are in U. A $CAS(u, v)$ on object X successfully writes v into X if and only if the value previously stored in X is equal to u. Otherwise, $CAS(u, v)$ does nothing to the value stored in X. Whether it succeeds or fails, $CAS(u, v)$ returns the old value of X. We present a formal way to describe the model we discussed here in Section 2.3.

2.2 Data Types

If we wish to use more complex data structures than those provided by the system, we must implement them in software. In order to describe the correctness of a concurrent implementation of a data type, Herlihy and Wing [24] defined a property called *linearizability*. It ensures that every operation on the concurrent objects appears to take effect atomically at some point between its invocation and response. Before we formally define linearizability, we introduce the definition of a sequential specification of a data type, adapted from [23, Chapter 9.4].

Definition 2.1. A data type is a tuple $\langle V, v_0, I, R, f\rangle$ consisting of

- a set V of values,
- an initial value $v_0 \in V$,
- a set I of invocations,
- a set R of responses, and
- a function f: $V \times I \to R \times V$.

Intuitively, an object of type $\langle V, v_0, I, R, f\rangle$ stores a value from V and starts with the initial value v_0. If an invocation $inv \in I$ is performed when the object's value is $v \in V$, then $f(v, inv) = (res, v')$ describes the outcome of the invocation: the object's value is changed to v' and the object returns the response res. For simplicity, we restrict data types to behave deterministically here. This covers most data types encountered in practice.

A data type can be manipulated by either one process or a set of processes. We define a sequential execution of a data type by only one process in Definition 2.2.

Definition 2.2. A *sequential execution* of a data type $\langle V, v_0, I, R, f\rangle$ is a finite sequence $\langle v_0, inv_1, res_1, v_1, inv_2, res_2, \cdots, v_k\rangle$, such that for all $0 \leq j < k$, $f(v_j, inv_{j+1}) = (res_{j+1}, v_{j+1})$.

We denote the empty sequence by ϵ and use $\diamond$ to denote concatenation of sequences. Using the definition of sequential executions of a data type, we can define the notion of a *sequential trace* of a data type. A *trace* is also called a *history* by Herlihy and Wing [24].

Definition 2.3. The *trace* of a sequential execution is defined inductively as follows:

- $trace(\epsilon) = \epsilon$, and
- $trace(\langle v, inv, resp\rangle \diamond E) = \langle inv, resp\rangle \diamond trace(E)$, for any execution $\langle v, inv, resp\rangle \diamond E$.

A sequence is a sequential trace of a data type if it is the trace of some sequential execution of that data type. Here, we give an example of a *fetch and increment* (*fetch&inc*) data type. A *fetch&inc* data type stores an integer value and provides only one type of operation. It atomically performs the following three small steps: (1) read the integer value of the object and store it to a local variable *val*; (2) increase the object's value by 1; (3) return *val*. More formally, the specification of a *fetch&inc* data type is shown in Definition 2.4.

Definition 2.4. A *fetch&inc* data type is defined as follows:

- $V = \mathbb{N}$,
- $v_0 = 0$,
- $I = \{fi\}$,
- $R = \{fiResp(n) \mid n \in \mathbb{N}\}$, and
- $f(v, fi) = (fiResp(v), v + 1)$, for any $v \in V$.

For instance, a *fetch&inc* data type may have a sequential execution:

$$E_s = \langle 0, fi, fiResp(0), 1, fi, fiResp(1), 2, fi, fiResp(2), 3, fi, fiResp(3), 4\rangle.$$

The trace T_s corresponding to E_s is:

$$T_s = \langle fi, fiResp(0), fi, fiResp(1), fi, fiResp(2), fi, fiResp(3)\rangle.$$

A data type can also be concurrently manipulated by a finite set of processes $PROC$. Its executions, known as *atomic* executions, are defined in Definition 2.5.

Definition 2.5. An *atomic execution* of a data type $\langle V, v_0, I, R, f\rangle$ manipulated by a set of processes $PROC$ is a finite sequence $\langle v_0, (inv_1, p_1), (res_1, p_1), v_1, (inv_2, p_2), (res_2, p_2), \ldots, v_k\rangle$ such that for all $0 \leq j < k$, $f(v_j, inv_{j+1}) = (res_{j+1}, v_{j+1})$ and for all $1 \leq j \leq k$, $p_j \in PROC$.

For instance, an atomic execution of a data type among processes p, q and r may look like:

$$E_a = \langle v_0, (inv, p), (res, p), v_1, (inv, r), (res, r), v_2, (inv, q), (res, q), v_3\rangle.$$

Or, we can put processes as subscripts for a more compact representation.

$$E_a = \langle v_0, inv_p, res_p, v_1, inv_r, res_r, v_2, inv_q, res_q, v_3\rangle.$$

The definition of the *atomic trace* can also be defined by applying Definition 2.3. Hence, *atomic traces* are the trace of atomic executions of a data type. For instance, the trace of E_a is:

$$T_a = \langle inv_p, res_p, inv_r, res_r, inv_q, res_q\rangle.$$

The invocation and the matching response by p compose a complete operation of p. For each invocation by process p, the matching response is the next response by p. Intuitively, in an atomic trace, each response of a process p is immediately preceded by a matching invocation of p.

An example of an atomic execution on a *fetch&inc* object manipulated by processes p, q, and r is shown below.

$$\langle 0, fi_p, fiResp_p(0), 1, fi_r, fiResp_r(1), 2, fi_p, fiResp_p(2), 3, fi_q, fiResp_q(3), 4\rangle.$$

Its atomic trace is:

$$\langle fi_p, fiResp_p(0), fi_r, fiResp_r(1), fi_p, fiResp_p(2), fi_q, fiResp_q(3)\rangle.$$

Besides sequential executions and atomic executions, there are concurrent executions of implementations of data types. We define concurrent executions and traces after introducing I/O automata.

2.3 Input/Output Automata

The *input/output (I/O) automaton* is a formal model for asynchronous computing [23]. It is a powerful and general model that is suitable for describing almost any type of asynchronous concurrent system. An I/O automaton is a simple type of state machine in which the transitions are associated with named *actions*, which are classified as *external* or *internal*. Following the definitions given by Lynch and Tuttle [11] and Doherty [5], external actions, which are visible to the outside world, are further classified according to whether they model *invocation* (*input*) or *response* (*output*) events. On the other hand, internal actions are visible only to the automaton itself. Given three disjoint sets *in, out*, and *int* of input, output and internal actions, respectively, we use the triple (in, out, int) as an *action signature* *S*. We denote the sets *in, out, and int* of the action signature S by $in(S), out(S)$, and $int(S)$, respectively. Furthermore, we define the *external actions*, $ext(S)$, to be $in(S) \cup out(S)$; $acts(S)$ to be all the actions of S.

Definition 2.6. An *input/output automaton* A consists of four components,

1. a set $states(A)$ of states,

2. a non-empty set $start(A) \subseteq states(A)$ of *start states*,

3. an action signature $sig(A)$, and

4. a transition relation $trans(A) \subseteq states(A) \times acts(sig(A)) \times states(A)$.

We use $acts(A)$ as shorthand for $acts(sig(A))$, and similarly for $in(A)$, and so on. We call an element (s, π, s') of $trans(A)$ a *transition* of A. The transition (s, π, s') is called an *input transition, output transition* or *internal transition*, based on whether the action π is an input action, output action or internal action. When $(s, \pi, s') \in trans(A)$, we write $s \xrightarrow{\pi}_A s'$, or $s \xrightarrow{\pi} s'$ when no confusion is possible. If $s \xrightarrow{\pi}_A s'$ we refer to s as the *pre-state* of the transition, and s' as the *post-state*. Now we present some definitions related to I/O automata, mainly adapted from Doherty [5].

Definition 2.7. For any I/O automaton A:

1. An *execution fragment* of A is a finite sequence $\alpha = \langle s_0, \pi_1, s_1, \pi_2, s_2, \cdots, \pi_k, s_k \rangle$ of alternating states and actions of A such that $(s_i, \pi_{i+1}, s_{i+1})$ is a transition of A for all $0 \leq i < k$. If such a finite execution fragment exists, we write $s_0 \stackrel{\alpha}{\Longrightarrow}_A s_k$, or $s_0 \stackrel{\alpha}{\Longrightarrow} s_k$ when no confusion is possible.

2. An *execution* of A is a finite execution fragment whose first state s_0 is in $start(A)$. We denote the set of executions of A by $execs(A)$.

3. For an execution α of A, $trace(\alpha)$ is the sequence α restricted to external actions of the automaton A.

4. The set $traces(A)$ is the set of traces of executions of A.

2.3.1 Concurrent Implementations and Linearizability

Intuitively, a *concurrent implementation* of a data type specifies a program that can be executed by each process $p \in PROC$ to perform an operation. This program will also specify the result of the operation to be returned to p. More formally, a concurrent implementation will be described as an I/O automaton, whose external actions are invocations and responses of the data type. Since the program that implements an operation may take many steps, those steps are all modelled as internal actions. When describing a concurrent implementation using an I/O automaton, we also put processes as subscripts of the actions in each transition to identify which process takes a step.

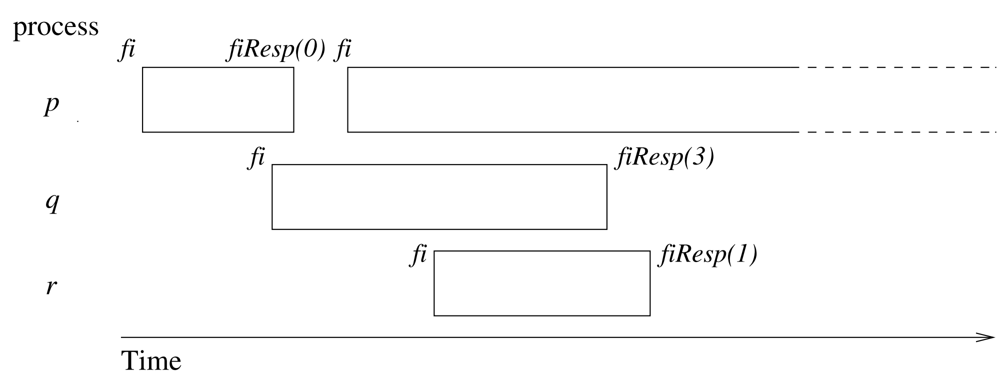
Thus, we can formally define concurrent executions and concurrent traces.

Definition 2.8. A *concurrent execution* of a concurrent implementation manipulated by a set of processes $PROC$ is an execution of its I/O automaton, in which each action has a process as its subscript.

Definition 2.9. A *concurrent trace* of a concurrent implementation manipulated by a set of processes $PROC$ is a trace of its I/O automaton.

Because the steps of processes are interleaved by the scheduler, an invocation of an operation by one process and its matching response are not guaranteed to be next to each other, in the trace of a concurrent implementation. Between an invocation and the matching response made by one process, there may be some invocations and responses for other processes. For example, if the set of processes is $\{p, q, r\}$, a concurrent trace of a data type $\mathcal{D}$ may look like:

$$T = \langle inv_p, inv_r, res_r, inv_q, res_p, res_q \rangle.$$



Trace T_1: $\langle fi_p, fi_q, fiResp_p(0), fi_p, fi_r, fiResp_q(3), fiResp_r(1) \rangle$

Figure 2.1: The trace from a finite execution sequence on $fetch\&inc$ object by three processes: p, q, and r.

Figure 2.1 shows one possible trace T_1 of a concurrent implementation of a *fetch*$\&$*inc* data type executed by processes p, q and r. The trace of a concurrent data type, like T_1, is complicated because of the interleaving of operations by different processes. Therefore, we need some properties to identify if a trace is "good" or not. Linearizability [24] is such a property which guarantees that each concurrent trace is equivalent to some legal atomic trace that satisfies its sequential specification. Before formally specifying linearizability, we need to introduce some definitions, adapted from [24].

Definition 2.10. Given a trace T and a process p, a *process subtrace*, $T|p$ (T at p), is the subsequence of all invocations and responses in T whose process names are p.

Definition 2.11. Two traces T and T' are *equivalent* if, for every process p, $T|p = T'|p$.

Definition 2.12. A trace T of a data type $\mathcal{D}$ is *well-formed* if, for each process p, its subtrace $T|p$ starts with an invocation, and alternates between invocations and responses.

We also define a partial order relation on operations of different processes. Recall that an operation e consists of the invocation $inv(e)$ together with the matching response $res(e)$ (if it exists).

Definition 2.13. The *irreflexive partial order* $<_T$ on the operations in trace T, is defined by: $e_i <_T e_j$ if and only if $res(e_i)$ precedes $inv(e_j)$ in T.

Informally, the irreflexive partial order $<_T$ shows a "sequential" relation among some operations. Pairs of operations that are not ordered by $<_T$ are regarded as "concurrent" operations. Straightforwardly, if a trace is atomic, $<_T$ becomes a total order. Based on the preceding definitions, we define a trace T to be *linearizable* as follows.

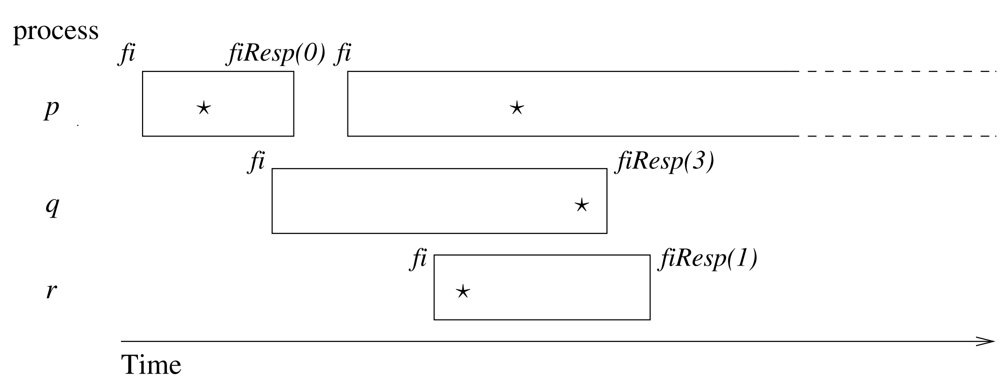
Definition 2.14. For a trace T', let $complete(T')$ be the maximal subsequence of T' consisting of all responses and their matching invocations. A trace T is *linearizable* with respect to a data type $\mathcal{D}$, if it can be extended (by appending zero or more response events) to some trace T' such that:

L1: $complete(T')$ is equivalent to some legal atomic trace S, and

L2: $<_{complete(T')} \subseteq <_S$.

A pending operation in an execution is an operation without a matching response. Intuitively, for each pending operation that takes effect but does not return a response, we add the response to obtain T'. For example, in Figure 2.1, the second operation made by process p is pending, but it must take effect in trace T_1, since that is the only way that process q can return $fiResp(3)$. On the other hand, $complete(T')$ excludes those pending operations that have not yet taken effect. As described by Herlihy [24], "L1 in Definition 2.14 states that processes act as if they were interleaved at the granularity of complete operations. L2 states that this sequential interleaving corresponds to the precedence ordering of operations." S is called a *linearization* of T. Note that L1 also implies that subtraces of each process in T are well-formed.

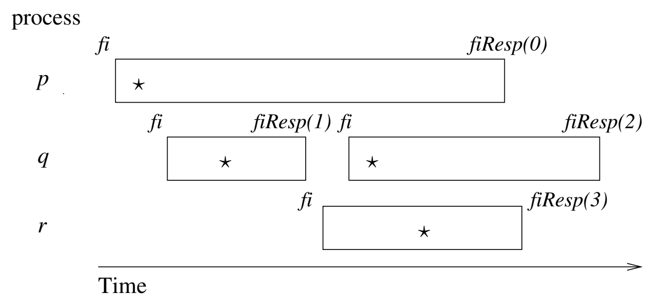
To show linearizability of a trace, we can identify a time within each operation when the operation can be considered to take effect, namely the *linearization point*, and show that ordering the operations in the concurrent execution by their linearization points gives an equivalent atomic trace. Each operation that has no response may or may not be assigned a linearization point. For example, we can assign linearization points to T_1 as shown in Figure 2.2 so that its corresponding legal atomic trace is T_0. Another example of assigning linearization points to a trace is illustrated in Figure 2.3(a). Since there is no way to assign linearization points to the trace shown in Figure 2.3(b), it is not linearizable. To see why, in any atomic trace that preserves the order $<_{complete(T)}$, there is a partial order relation between the first operations performed by process q and r (because $fiResp_q(2)$ precedes fi_r in T_3). However, their responses violate the specification of the *fetch&inc* data type.



Trace T_1: $\langle fi_p, fi_q, fiResp_p(0), fi_p, fi_r, fiResp_q(3), fiResp_r(1)\rangle$

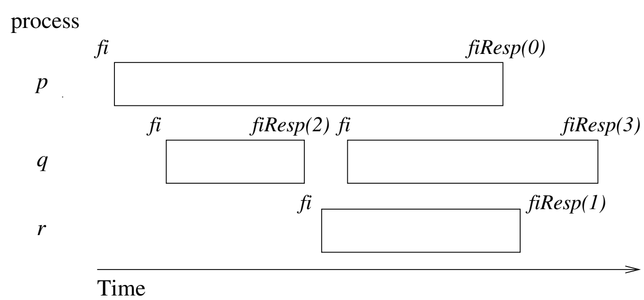
Figure 2.2: Trace T_1 of a *fetch&inc* object with linearization points (shown as "stars"). Here, p's second operation took effect but the operation was pending.

Definition 2.15. A *linearizable implementation* of a data type $\mathcal{D}$ is one whose concurrent traces are linearizable.



Trace T_2:$\langle fi_p(), fi_q(), fiResp_q(1), fi_r(), fi_q(), fiResp_p(0), fiResp_r(3), fiResp_q(2)\rangle$

(a)



Trace T_3:$\langle fi_p(), fi_q(), fiResp_q(2), fi_r(), fi_q(), fiResp_p(0), fiResp_(1), fiResp_q(3)\rangle$

(b)

Figure 2.3: (a) shows a possible way to assign linearization points (shown as "stars") in trace T_2. (b) illustrates an execution that is not linearizable.

In other words, a concurrent data structure is linearizable if, for each concurrent trace of the data type, there is an atomic trace in which every operation returns the same result, and non-concurrent operations occur in the same order in the atomic trace as in the concurrent trace.

Definition 2.16. The trace inclusion relation $\subseteq_T$ is defined as follows: for any I/O automata A and B, $A \subseteq_T B$ *iff* $traces(A) \subseteq traces(B)$.

An I/O automaton can be viewed as a "black box" from the point of view of a user. What the user sees is just the trace of the automaton's execution. If, for any automata A and B, $A \subseteq_T B$, then any (external) behaviour exhibited by A could also be exhibited by B. Trace inclusion allows us to identify if one I/O automaton specifies the desired external behaviour of another automaton. Section 2.4 describes a formal verification technique based on the trace inclusion property between a specification automaton and an implementation automaton.

2.3.2 Canonical Automata

Doherty [5] described a *canonical automaton* which is able to capture all linearizable traces of the data type $\mathcal{D}$. Doherty proved the traces of *any* automaton that is a linearizable implementation of $\mathcal{D}$ will be included in the traces of the canonical automaton [5].

Recall that an automaton consists of four key parts: *states*, *start states*, *actions* and a *transition relation*. In the canonical automaton CA for a data type $\mathcal{D} = (V, v_0, I, R, f)$ and a set of processes $PROC$, the state consists of a value from V and a value of the program counter for each process. Intuitively, a state records the current value of an instance of $\mathcal{D}$, and the value of the program counter for process p indicates the next action that p is allowed to perform. $Start(CA)$ contains a single state where the value is the initial value v_0 of $\mathcal{D}$ and program counter of each process p is set to be *idle*, indicating that p is ready to perform an invocation. As defined in [5], the canonical automaton's external input actions are invocations of the data type coupled with processes. Similarly, each external output action is one of the data type's responses together with a process. Thus, we will have the following external signature for CA.

- $in(CA) = \{inv_p \mid inv \in I, p \in PROC\}$
- $out(CA) = \{resp_p \mid resp \in R, p \in PROC\}$

We call $in(CA)$ its invocations, and $out(CA)$ its responses. Internal actions in $int(CA)$ which represent the linearization points, apply the update function f of $\mathcal{D}$'s specification to the value. The transition relation $trans(CA)$ is constructed straightforwardly based on *states* and *actions*.

Figure 2.4 shows how we construct $states(CA)$, $start(CA)$, $in(CA)$, $out(CA)$ and $int(CA)$, given a set of processes *PROC*. To guarantee the well-formed property of the traces of *CA*, there are three types of values for the program counter: *pc_idle*, *pc_inv* and *pc_resp*. Intuitively, *pc_idle* indicates that only invocations are allowed to be performed, and *pc_inv* indicates that a process has performed the invocation *inv* and is able to execute an internal action. Similarly, *pc_resp* indicates that the response *resp* is allowed to be returned to a process.

Because a *state* of an automaton is usually represented using a Cartesian product, we introduce "*state variables*" [5] to describe each component of it. For example, we use a state variable "pc_p" for each $p \in PROC$ and a state variable "val", where $\mathrm{pc}_p = \pi_p(\pi_2(s))$ and $\mathrm{val} = \pi_1(s)$, for a state $s \in states(CA)$. Thus, we have a more

$states(CA) = V \times \prod_{p \in PROC} Pcval$, where $Pcval = \{pc\_idle\} \cup \{pc\_inv : inv \in I\} \cup \{pc\_resp : resp \in R\}$

$start(CA) = \{v_0\} \times \prod_{p \in PROC} \{pc\_idle\}$, where v_0 is the initial value of $\mathcal{D}$

$in(CA) = \{inv_p \mid inv \in I, p \in PROC\}$

$out(CA) = \{resp_p \mid resp \in R, p \in PROC\}$

$int(CA) = \{do\_inv_p \mid inv \in I, p \in PROC\}$

Figure 2.4: The *states*, *start* and *actions* of the canonical automaton CA with respect to a data type $\mathcal{D}$ and a set of processes $PROC$.

convenient way to describe $trans(CA)$.

Based on the construction of $states(CA)$, $start(CA)$ and $acts(CA)$, the transition relation $trans(CA) \subseteq states(CA) \times acts(CA) \times states(CA)$ of the canonical automaton is obvious. We describe $trans(CA)$ as the set of all triples (s, π, s') such that the state s satisfies the some preconditions before executing the action π and the state s' is obtained from s by updating s according to π. When a process's program counter is *pc_idle*, it is able to perform an invocation. After the invocation, it may perform an internal action and then a response action may be invoked eventually. Whenever an action is performed by p, its program counter is updated to ensure the well-formed property. Additionally, when an internal action is executed, the value of $\mathcal{D}$ in the state of CA is also updated according to the update function f of $\mathcal{D}$. Table 2.1 illustrates how we use **Pre** and **Eff** to capture the *pre-conditions* before executing an action and the *effects* of the action on the state, respectively. Note that the effects of each action form a set of parallel assignments. State variables that are not mentioned in **Eff** remain the same.

Action	**Pre**	**Eff**
inv_p	$\mathrm{pc}_p = pc\_idle$	$\mathrm{pc}_p \leftarrow pc\_inv$
do_inv_p	$\mathrm{pc}_p = pc\_inv$ $\mathrm{val} = v$	$\mathrm{val} \leftarrow v'$, $\mathrm{pc}_p \leftarrow pc\_resp$, *where* $(resp, v') = f(v, inv)$
$resp_p$	$\mathrm{pc}_p = pc\_resp$	$\mathrm{pc}_p \leftarrow pc\_idle$

Table 2.1: Transitions of a Canonical Automaton.

In Table 2.1, inv_p, do_inv_p and $resp_p$ all represent actions, where $inv \in I$, $resp \in R$ and $p \in PROC$. The construction of our canonical automaton follows Doherty [5]. A slightly different construction is given by [23, Section 13.2]. To show a concrete example of constructing a canonical automaton, recall the $fetch\&inc$ data type described in Definition 2.4. The canonical automaton CF for the *fetch&inc* data type is shown in Figure 2.5 and Table 2.2.

Linearizability of the Canonical Automaton

This section proves that all traces of the canonical automaton are well-formed and linearizable. The results in this section are fairly straightforward and mainly follow

$states(CF) = \mathbb{N} \times \prod_{p \in PROC} Pcval$, where $Pcval = \{pc\_idle, pc\_fi\} \cup \{pc\_fiResp(n) \mid n \in \mathbb{N}\}$

$start(CF) = \{0\} \times \prod_{p \in PROC} \{pc\_idle\}$

$in(CF) = \{fi_p \mid p \in PROC\}$

$out(CF) = \{fiResp(n) \mid n \in \mathbb{N}, p \in PROC\}$

$int(CF) = \{do\_fi_p \mid p \in PROC\}$

Figure 2.5: *states*, *start* and *actions* of the canonical automaton CF for the *fetch&inc* data type.

Action	**Pre**	**Eff**
fi_p	$\mathrm{pc}_p = pc\_idle$	$\mathrm{pc}_p \leftarrow pc\_fi$
$doFi_p$	$\mathrm{pc}_p = pc\_fi$ $\mathrm{val} = v$	$\mathrm{pc}_p \leftarrow pc\_fiResp(v)$ $\mathrm{val} \leftarrow v + 1$
$fiResp_p(n)$	$\mathrm{pc}_p = pc\_fiResp(n)$	$\mathrm{pc}_p \leftarrow pc\_idle$

Table 2.2: Transitions for the canonical automaton CF of the *fetch&inc* data type.

the proofs in [23] and [5]. We assume we have a data type $\mathcal{D} = (V, v_0, I, R, f)$ and the canonical automaton CA as described in the preceding construction.

Recall that a *linearizable* I/O *automaton* A should satisfy three properties: its external actions must match the invocations and responses of $\mathcal{D}$, its traces must be *well-formed*, and its traces must be *linearizable*. According to the construction, it is trivial to show that $in(CA)$ and $out(CA)$ correspond to *invocations* and *responses* of $\mathcal{D}$.

Lemma 2.17. *All the traces of CA are well-formed.*

Proof. Consider an execution of CA and a process p.

- For the state $s \in start(A)$, $pc_p = pc\_idle$. Because only p's actions modify pc_p, the first *p-indexed* action has to be an invocation according to the transitions in Table 2.1.

- Immediately after any *p-indexed* invocation inv_p, $pc_p = pc\_inv$. According to the Table 2.1, the only admissible action for p is a do_inv_p, if it exists.

- The next *p-indexed* action after each do_inv_p is a response, if it exists. The *p-indexed* response action also sets pc_p to pc_idle.

- The next *p-indexed* action after each response $resp$ is an invocation, if it exists, because $pc_p = pc\_idle$.

□

We next show that all traces of CA are linearizable.

Lemma 2.18. *All traces of CA are linearizable.*

Proof. Let $\alpha = \langle s_0, \pi_1, s_1, \pi_2, s_2, \cdots, s_{n-1}, \pi_n, s_n \rangle$ be an execution of CA and T be the trace of α. A complete operation in α consists of its invocation *inv_op*, the internal action *do_inv_op*, and the matching response *resp_op*.

Let T' be an extension of T obtained by appending the response for every pending operation in T that has a *do_inv* in α. Let $op_1, op_2, op_3, \cdots, op_k$ $(k \leq n)$ be the operations whose *do_inv* actions are in α, in the order of those *do* actions. Let S be the atomic trace $\langle inv\_op_1, resp\_op_1, inv\_op_2, resp\_op_2, \cdots, inv\_op_k, resp\_op_k \rangle$.

According to Definition 2.14, we must show that S is a legal trace with respect to the data type $\mathcal{D} = (V, v_0, I, V, f)$; $<_{complete(T')} \subseteq <_S$; and $complete(T')$ is equivalent to S.

(a). Let the value of the state just prior to the *do_inv* action of op_i in α be v_{i-1}. Since only *do_inv* actions modify the value component of a state of CA, v_i remains the same among all states between states after do_inv_{i-1} and before do_inv_i. Moreover, v_0 is the same as the initial value in state s_0. Thus, by the definition of *do_inv* action of op_i, $f(v_{i-1}, inv\_op_i) = (resp\_op_i, v_i)$, where in the state after $do\_inv\_op_i$, its $pc_p = pc\_resp$. Thus, $resp\_op_i = resp$. So, $\forall i : f(v_{i-1}, inv\_op_i) = (resp\_op_i, v_i)$ and S is a trace of a legal execution.

(b). For all i, j, if $resp\_op_i$ precedes inv_op_j in $complete(T')$, op_i precedes op_j in S. Because we have shown in the proof of Lemma 2.17 that the *do* action of an operation is in between its invocation and matching response, do_inv_i precedes *do_inv*$_j$ in α. Therefore, by the construction of S, *resp_op*$_i$ precedes *inv_op*$_j$ in S. So we have $<_{complete(T')} \subseteq <_S$.

(c). We argue that $S|p$ is the same as $complete(T')|p$ for every process p. As proved in part (b), we can easily obtain $<_{complete(T')|p} \subseteq <_{S|p}$, where both $<_{complete(T')|p}$ and $<_{S|p}$ are total orders. Combined with the fact that for every operation, its invocation precedes the matching response, the order of invocations and responses in $complete(T')|p$ and $S|p$ must be the same. It remains to show that every operation in $complete(T')$ if and only if it is in S. This is true because any operation, either a complete operation or a pending one, which contains the *do* action is in $complete(T')$, and also is in S by the construction. Pending operations which do not have *do* actions are in neither $complete(T')$ nor S.

$\square$

Lynch [23] shows a proof of Lemma 2.18 for a slightly different canonical automaton, but the idea of that proof is similar to the one given here. The intuition in both proofs is that we can always linearize the traces according to the *do_inv* actions of the canonical automaton. Since *do_inv* actions directly follow the update function of the data type, this order of execution sequence forms a legal sequential execution.

2.4 Simulations

The *simulation* method [12] is an approach for proving one concurrent system A implements B by showing a trace inclusion relationship between them. In a simulation proof, each system is modelled as an I/O automaton and we show that each transition in A has a corresponding execution in B, such that their traces are the same. This technique has been frequently used for formal verification of linearizability of concurrent implementations. An implementation is linearizable if the traces of the automaton that models the implementation of data type $\mathcal{D}$, are subsumed by the traces of the canonical automaton of $\mathcal{D}$. To show the inclusion relationship, we mainly consider two types of simulations: forward and backward.

2.4.1 Forward Simulations

A $forward\ simulation$ [12] from automaton A to automaton B is a relation *fsr* from states of A to states of B such that every initial state of A is related to an initial state of B, and every action of A yields a corresponding sequence of actions of B.

Definition 2.19. A *forward simulation* from the I/O automaton A to I/O automaton B is a relation $fsr \subseteq states(A) \times states(B)$ that satisfies the following properties:

1. For every $s \in start(A)$, there exists a $u \in start(B)$, such that $(s, u) \in fsr$.

2. If $s \xrightarrow{\alpha}_A s'$ and $(s, u) \in fsr$, then there exists $u \overset{\hat{\alpha}}{\Longrightarrow}_B u'$ for some u' such that $(s', u') \in fsr$, and

3. the external action in $\hat{\alpha}$ is the same as the external action in α (*i.e.*, either equals α, if α is an external action, or is empty otherwise).

Recall that $s \xrightarrow{\alpha}_A s'$ denotes that by performing the action α, state s becomes the post state s' in A. The notation $u \overset{\hat{\alpha}}{\Longrightarrow}_B u'$ means that in B, the automaton moves from state u to u' by performing a sequence of actions $\hat{\alpha}$. If the relation *fsr* over *states*(A) and *states*(B) in Definition 2.19 is a function, we call it a *refinement* [12]. A refinement is a simplified forward simulation that is often used in formally verifying the correctness of concurrent implementations [5, 7, 9, 10, 13, 16].

Theorem 2.20. *If fsr is a forward simulation from A to B, then traces(A) $\subseteq$ traces(B) [12].*

Proof. Let $E_A = \langle s_0, \pi_1, s_1, \pi_2, \cdots, s_{n-1}, \pi_n, s_n \rangle$ be an execution of A and T_A be the trace of E_A. We argue that there exists an execution E_B of B that has the same trace as T_A. Let c_0 be an initial state of B, such that $(s_0, c_0) \in fsr$ (such a c_0 exists according to Definition 2.19). We do induction on the length of E_A. If we know $s_0 \xrightarrow{\langle \pi_1, \cdots, \pi_i \rangle}_A s_i$ and $c_0 \overset{\langle \hat{\pi}_1, \cdots, \hat{\pi}_i \rangle}{\Longrightarrow}_B c_i$ such that $c_0 \in fsr(s_0)$, $c_i \in fsr(s_i)$ and $trace(\langle \pi_1, \cdots, \pi_i \rangle) = trace(\langle \hat{\pi}_1, \cdots, \hat{\pi}_i \rangle)$, we can construct an execution sequence $c_i \overset{\hat{\pi}_{i+1}}{\Longrightarrow}_B c_{i+1}$ of B for $s_i \xrightarrow{\pi_{i+1}}_A s_{i+1}$ of A, where $trace(\pi_{i+1}) =$

$trace(\hat{\pi}_{i+1})$ and $c_{i+1} \in fsr(s_{i+1})$. Then, $c_0 \xRightarrow{\langle\hat{\pi}_1,\cdots,\hat{\pi}_{i+1}\rangle}_B c_{i+1}$, $c_{i+1} \in fsr(s_{i+1})$ and $trace(\langle\hat{\pi}_1, \cdots, \hat{\pi}_{i+1}\rangle) = trace(\langle\pi_1, \cdots, \pi_{i+1}\rangle)$. The base case is trivial when $i = 0$. Finally, we have $trace(\langle\hat{\pi}_1, \cdots, \hat{\pi}_n\rangle) = trace(\langle\pi_1, \cdots, \pi_n\rangle) = trace(E_A)$. □

An Example of a Forward Simulation

We shall show an example of how a forward simulation can be used to prove the correctness of an implementation. A simple implementation of the *fetch&inc* using a *CAS* object is illustrated in Figure 2.6. Recall the specification $\mathcal{D} = (V, v_0, I, R, f)$ of a $fetch\&inc$ data type in Definition 2.4 and the canonical automaton CF in Figure 2.5 and Table 2.2. It is fairly easy to see that this implementation is linearizable: the linearization point of each *fetch&inc* operation is when it performs its successful *CAS*. We shall formalize this argument using a forward simulation.

As described in Section 2.1, a *CAS(X,u,v)* operation always returns the old value stored in X. It successfully changes the value stored in X to v if and only if the old value of X is equal to u. Otherwise, it does not change the value. In order to show the relation between the implementation and its specification, we first formalize the algorithm in Figure 2.6 as a concrete automaton C. Let $PROC$ be a set of process and let $Pcval = \{pc\_idle, pcLine1, pcLine2\} \cup \{pc\_fiResp(n) \mid n \in \mathbb{N}\}$. Consider the concrete automaton for the *fetch&inc* implementation shown in Figure 2.7.

The state of the concrete automaton consists of a shared variable v, a local vari-

Fetch&Inc()

while TRUE {

$res \leftarrow v$

if $CAS(v, res, res+1) = res$

return res

}

Figure 2.6: An algorithm that uses CAS to implement the *fetch-and-increment* data type.

$$states(C) = \{v \mid v \in \mathbb{N}\} \times \prod_{p \in PROC} \{res \mid res \in \mathbb{N}\} \times \prod_{p \in PROC} Pcval$$

$$start(C) = \{s \in states(C) \mid s.v = 0 \wedge \forall p \in PROC : s.res_p = 0 \wedge s.pc_p = idle\}$$

$$in(C) = \{fi_p \mid p \in PROC\}$$

$$out(C) = \{fiResp_p(n) \mid n \in \mathbb{N}, p \in PROC\}$$

$$int(C) = \{line1_p, line2T_p, line2F_p\}$$

Figure 2.7: The concrete automaton that models the implementation of the *fetch-and-increment* data type.

able res_p for each p and a program counter pc_p for each p. Each action corresponds to executing a line of the code. For simplicity, res_p is initialized to 0 for all p. We use multiple actions to model those lines of code which may subsequently execute different lines depending on their pre-condition, such as **if**, **until** and **while** operations. For example, in this algorithm, Line 2 in Figure 2.6 is modelled as two internal actions: $line2T_p$ and $line2F_p$. One indicates the **if** condition on *line* 2 evaluates to *true* and the other indicates it evaluates to *false*.

Action	**Pre**	**Eff**
fi_p	$\mathrm{pc}_p = pc\_idle$	$\mathrm{pc}_p \leftarrow pcLine1$
$line1_p$	$\mathrm{pc}_p = pcLine1$	$\mathrm{pc}_p \leftarrow pcLine2$ $\mathrm{res}_p \leftarrow v$
$line2F_p$	$\mathrm{pc}_p = pcLine2$ $v \neq \mathrm{res}_p$	$\mathrm{pc}_p \leftarrow pcLine1$
$line2T_p$	$\mathrm{pc}_p = pcLine2$ $v = \mathrm{res}_p$	$\mathrm{pc}_p \leftarrow pc\_fiResp(\mathrm{res}_p)$ $v \leftarrow \mathrm{res}_p + 1$
$fiResp_p(n)$	$\mathrm{pc}_p = pc\_fiResp(n)$	$\mathrm{pc}_p \leftarrow pc\_idle$

Table 2.3: Transitions of the concrete automaton C of the *fetch-and-increment* algorithm.

The preconditions and effects of each action are shown in Table 2.3. The internal action $line1_p$ formalizes the "read" action in Figure 2.6, and $line2F_p$ captures a failed CAS (where the value it wants to change is not equal to its expected value). Action $line2T_p$ corresponds to a successful CAS, where the value is increased by one. After defining the concrete automaton of the algorithm, we need to show that there is a forward simulation from the concrete automaton C to canonical automaton CF, to show that C implements CF. To show that a forward simulation exists, we establish a relation *fsr* over $states(C)$ and $states(CF)$ and an action correspondence between $acts(C)$ and $acts(CF)$. First of all, because the program counters are one of the most important components of the states for both automata, we present how they are changed and their relation in C and CF in Figure 2.8.

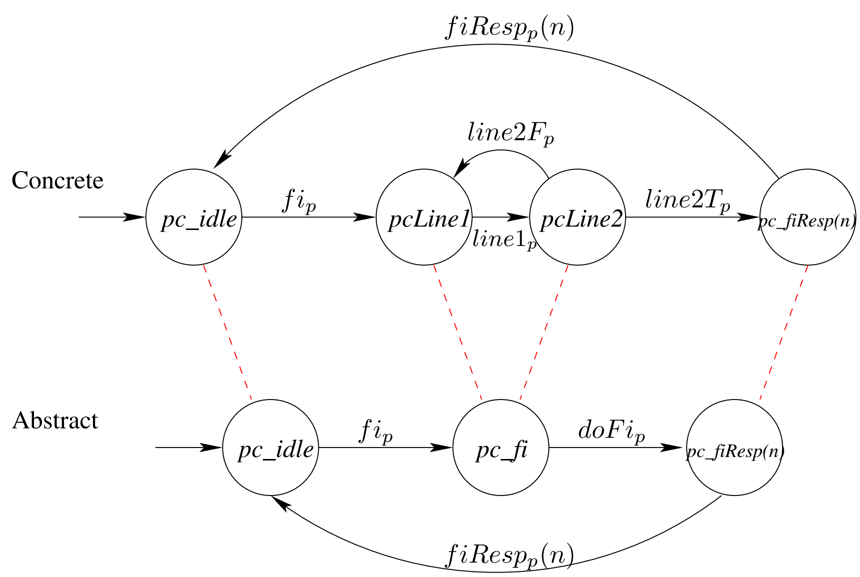


Figure 2.8: The state diagrams of process p's program counter value for both concrete and canonical automata and the relation between them.

Intuitively, the connection of program counters between C and CF relates to the action correspondence of the two automata. A successful CAS action in C corresponds to a $doFi_p$, which increases the value by one in CF. Any other internal actions in C corresponds to a null sequence of actions in CF. Additionally, all external actions are performed in the same way in C and CF in order to have the same traces in the forward simulation. This gives us the forward simulation relation in terms of the values of the program counter of process p, illustrated by the red dotted lines in Figure 2.8. Together with the requirement that the value of the *fetch&inc* data type recorded in both automata is the same, we have the forward simulation relation $fsr(s, s')$, where $s \in states(C)$ and $s' \in states(CF)$, defined as follows:

$$fsr(s, s') = (s.v = s'.val) \wedge \tag{2.1}$$

$$\Big(\forall p \in PROC : (s.pc_p = s'.pc_p) \vee \tag{2.2}$$

$$(s'.pc_p = pc\_fi \wedge (s.pc_p = pcLine1 \vee s.pc_p = pcLine2))\Big). \tag{2.3}$$

Basically, (2.1) requires the value of the abstract data type to match the value stored in *CAS* object v. (2.2) and (2.3) require that the values of the program counters are also the same, except that the two values, $pcLine1$ and $pcLine2$, in C both correspond to a single value pc_fi in CF. Because local variables are invisible from CF, they may be arbitrary. Intuitively, this mapping captures the fact that internal actions: $line1$ and $line2F$ correspond to null actions in CF.

Lemma 2.21. *There exists a forward simulation from C to CF using the relation* *fsr defined above. (We also proved this lemma using the PVS theorem prover.)*

Proof. We prove that (2.1)-(2.3) define a forward simulation, according to the Definition 2.19. Firstly, if $s \in start(C)$ then there exists a $u \in start(CF)$ such that $u.v = s.val$ and $\forall p : u.pc_p = s.pc_p = pc\_idle$. Thus $(s, u) \in fsr$.

Secondly, if $s \xrightarrow{\alpha}_C s'$ and $(s, u) \in fsr$, we show there exists a state u' and a sequence of actions $\hat{\alpha}$ such that $u \overset{\hat{\alpha}}{\Longrightarrow}_{CF} u'$ and $(s', u') \in fsr$, and the external actions in $\hat{\alpha}$ are the same as the external actions in α. Because there are five types of action: $fi_p, line1_p, line2T_p, line2F_p$ and $fi\_Resp_p(n)$ in C, we prove this property by distinguishing the following cases.

1. If $\alpha = fi_p$, let $u' \in states(CF)$, such that $u'.v = u.v$ and $\forall q \neq p : u'.pc_q = u.pc_q, u'.pc_p = pc\_fi$. We then have $u \xrightarrow{fi_p}_{CF} u'$ in CF according to Table 2.3. Because $s'.pc_p = pc\_fi$ and $(s, u) \in fsr$, we know $(s', u') \in fsr$.

2. If $\alpha = fiResp_p(n)$, let $u' \in states(CF)$, such that $u'.v = u.v$ and $\forall q \neq p : u'.pc_q = u.pc_q, u'.pc_p = pc\_idle$. We then have $u \xrightarrow{fiResp(v)}_{CF} u'$ according to Table 2.3. Because $s'.pc_p = pc\_idle$ and $(s, u) \in fsr$, we know $(s', u') \in fsr$.

3. If $\alpha = line2T_p$, let $k = s.v = u.val$ and $u' \in states(CF)$ such that $\forall q \neq p : u'.pc_q = u.pc_q$ and $u'.pc_p = pc\_fiResp(k), u'.val = k + 1$. We then have $u \xrightarrow{doFi_p}_{CF} u'$, since $u.pc_p = pc\_fi$ and $u.val = k$ according to $(s, u) \in fsr$. Because $s'.pc_p = pc\_fiResp(k), u'.val = s'.v = k+1$ and $(s, u) \in fsr$, we know $(s', u') \in fsr$.

4. If $\alpha = line1_p$ or $\alpha = line2F_p$, let $u' \in states(CF)$ such that $u' = u$. We have $u \xrightarrow{nil}_{CF} u'$. We easily obtain $(s', u') \in fsr$, because for $line1_p$: $s'.pc_p = pc\_Line2$ and $u'.pc_p = pc\_fi$, or for $line2F_p$: $s'.pc_p = pc\_Line1$ and $u'.pc_p = pc\_fi$.

$\square$

2.4.2 Backward Simulations

A *backward* simulation relation [12] *bsr* is similar to a forward simulation relation, but the main difference is that in a forward simulation, we reason about execution sequences in a forward direction and in a backward simulation from A to B, we start from the end of A's execution and construct the corresponding execution of B backwards.

Definition 2.22. A backward simulation from the I/O automaton A to the I/O automaton B is a total relation[1] $bsr \subseteq states(A) \times states(B)$ that satisfies:

1. If $s \in start(A)$ and $u \in states(B)$, for all $(s, u) \in bsr$, $u \in start(B)$, and

2. if $s' \xrightarrow{\alpha}_A s$ and $u \in states(B)$ such that $(s, u) \in bsr$, then there exists a state $(s', u') \in bsr$ such that $u' \stackrel{\hat{\alpha}}{\Longrightarrow}_B u$, and

[1]A relation R over $states(A)$ and $states(B)$ is total if, for every $a \in states(A)$, there exists $b \in states(B)$ such that $(a, b) \in R$ is *true*.

3. the external action in $\hat{\alpha}$ is the same as the external action in α (*i.e.*, either equals α, if α is an external action, or is empty otherwise).

Theorem 2.23. *If bsr is a backward simulation from A to B, then traces(A) $\subseteq$ traces(B).*

Proof sketch. The idea of the proof here is similar to the proof of Theorem 2.20. It also can be found in [12]. The intuition is that given an execution $\alpha = \langle s_0, \pi_1, s_1, \cdots, \pi_n, s_n \rangle$ in A, we construct a corresponding execution $\hat{\alpha}$ in B starting from the end to the beginning inductively. Because the states $u_{i+1} \in states(B)$ and $s_{i+1} \in states(A)$ are related by bsr, there is a $u_i \in states(B)$ such that $(u_i, s_i) \in bsr$ and $u_i \overset{\hat{\pi}_i}{\Longrightarrow}_B u_{i+1}$ and $trace(\pi_i) = trace(\hat{\pi}_i)$ according to the definition of backward simulation. For $s_0 \in start(A)$, by property 1 of Definition 2.22, we have a $u_0 \in start(B)$ such that $u_0 \xRightarrow{\langle \hat{\pi}_1, \cdots, \hat{\pi}_n \rangle}_B u_n$ and $trace(\langle \pi_1, \cdots, \pi_n \rangle) = trace(\langle \hat{\pi}_1, \cdots, \hat{\pi}_n \rangle)$. $\square$

Intuitively, backward simulations are similar to forward simulations, except that in a backward simulation, *all* states in the image of a state in $start(A)$ are in $start(B)$, whereas, in a forward simulation, *some* states in the image of $start(A)$ are in $start(B)$. This is because when we construct a related trace backwards, the first state of the trace should be an initial state.

Both forward and backward simulations can be used to show one automaton implements another. People use backward simulations because sometimes it is more intuitive to show a backward simulation relation between two automata [7, 13, 15].

We use a backward simulation in Chapter 4.

Sometimes, to show that an automaton C implements another automaton A, we create an intermediate automaton B and show a forward simulation from C to B and a backward simulation from B to A. Together, these imply $traces(C) \subseteq traces(A)$. This approach is called a hybrid forward and backward simulation.[12]

3 Non-blocking Binary Search Trees and a Simplified Algorithm

A binary search tree (BST) [25] is one of the most fundamental data structures used in the traditional sequential setting. It can be used to support sorting and searching algorithms and also to implement sets, multisets, priority queues and dictionaries.

A node in a BST with or without children is called an *internal* node or a *leaf*, respectively. The node without a parent is the *root*. Each internal node can have two children *left* and *right*. Every node stores a *key*. Node x is a *descendant* of node y, if x is the child of y or x is a descendant of y's child. Intuitively, if there is a path of child pointers from y to x, x is a descendant of y. Figure 3.1 illustrates an instance of a BST, whose nodes store integer keys. The node with key 10 is the *root*. The node containing key 4 is one of the descendants of the node with key 6.

The subtree of a BST rooted at a given node is the tree containing that node and all of its descendants. For example, the nodes with keys $6, 3, 4$ and 8 form a subtree of the BST shown in Figure 3.1. A BST must also have an important property in

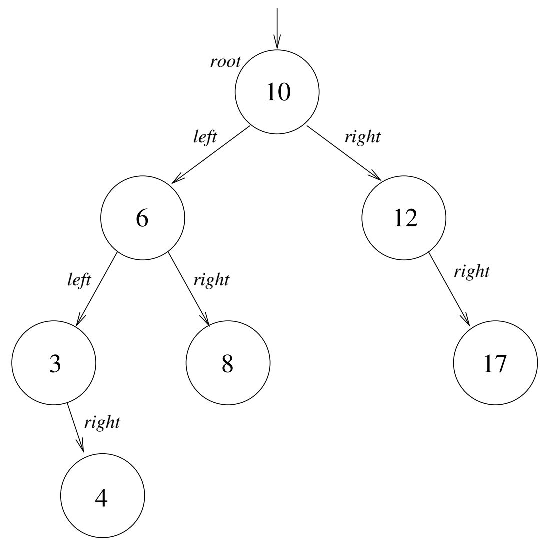


Figure 3.1: A typical binary search tree with integers as key values.

terms of its key values: for every internal node x, all keys in the left subtree of that node are less than $x.key$ and all keys in the right subtree are greater than or equal to $x.key$.

A BST can be used to implement a *set* data type, which stores a set of keys and provides *find*, *insert* and *delete* as basic operations.

Definition 3.1. A set data type $\mathcal{SET}$ has the following sequential specification:

- a state set $S = P(Key)$, where Key is a totally ordered set of all possible keys,
- an initial value $s_0 = \emptyset$,
- a set $I = \{FindInv(k), InsertInv(k), DeleteInv(k) \mid k \in Key\}$ of invocations,

- a set $R = \{FindResp(r), InsertResp(r), DeleteResp(r) \mid r \in boolean\}$ of responses, and

- an update function $f : V \times I \rightarrow R \times V$, such that:

$$\begin{cases} f(s, findInv(k)) = (findResp(true), s), \text{if } k \in s, \\ f(s, findInv(k)) = (findResp(false), s), \text{if } k \notin s, \\ f(s, insertInv(k)) = (insertResp(false), s), \text{if } k \in s, \\ f(s, insertInv(k)) = (insertResp(true), s \cup \{k\}), \text{if } k \notin s, \\ f(s, deleteInv(k)) = (deleteResp(true), s - \{k\}), \text{if } k \in s, \\ f(s, deleteInv(k)) = (deleteResp(false), s), \text{if } k \notin s, \end{cases}$$

  for all $k \in Key$ and $s \in S$.

Intuitively, *find* operations return *true* or *false* depending on whether the given key value is in the set or not. An *insert* operation inserts a new key into the set and returns *true* if the key was not already in it. The operation returns *false* and the set remains unchanged if the given key is already in the set. (We assume that the set data type does not allow duplicate keys.) A *delete* operation removes the given key from the set and returns *true* if the key is in the set. Otherwise, it return *false* and the set remains unchanged.

3.1 A Non-blocking Binary Search Tree Algorithm

Ellen et al. [3] developed the first efficient non-blocking implementation of a BST for an asynchronous shared-memory system. They provided a detailed proof of correctness, which was written in natural language. The BST algorithm they considered is *leaf-oriented*, meaning that all keys in the set are stored in leaf nodes and each internal node has exactly two children. Internal nodes only store auxiliary keys that are used to direct the searches towards the leaf containing a particular key.

Definition 3.2. Given a key k, the *search path* for k in a leaf-oriented BST is the sequence of nodes $\langle n_0, n_1, n_2, \cdots, n_m \rangle$, such that n_0 is the root, n_m is a leaf, and for $1 \leq i \leq m$, n_i is the left child or right child of n_{i-1} depending on whether $k < n_{i-1}.key$ or $k \geq n_{i-1}.key$, respectively.

3.1.1 Implementation Overview

To support the set data type, the non-blocking BST provides algorithms for three operations: *find*, *insert* and *delete*. All of them use a common sub-routine called *search*, which starts from *root* and searches toward a leaf that potentially contains the given key. The *find* operation returns *true* if the leaf node where the *search* terminates contains the given key. Otherwise, it returns *false*. Examples of a successful and unsuccessful *find* operation are shown in Figure 3.2. Square boxes and circles represent leaf nodes and internal nodes, respectively. Triangles represent subtrees.

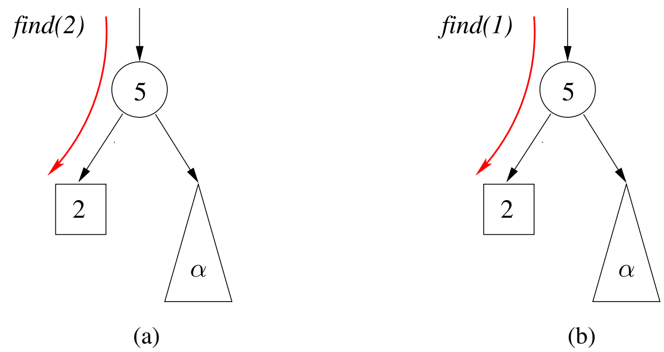


Figure 3.2: An example of *find* operations in a leaf-oriented BST. (a). The $find(2)$ operation ends with a node containing key 2, and returns $true$. (b) The $find(1)$ operation ends up with a node containing key 2, and returns $false$.

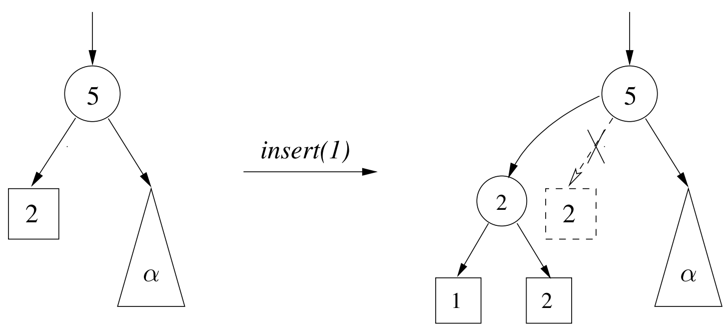


Figure 3.3: An example of an *insert* operation in a leaf-oriented BST.

A typical successful *insert* and *delete* operation on a leaf-oriented BST are shown in Figure 3.3 and 3.4. The $insert(1)$ operation locates a leaf node which potentially contains key 1 by using the *search* subroutine. If it successfully finds such a node, $insert(1)$ returns *false* since no duplicated keys are allowed. In Figure 3.3, because the leaf does not contain key 1, the *search* tries to insert key 1 into the BST by replacing the leaf with a subtree containing three nodes. Two leaves containing the

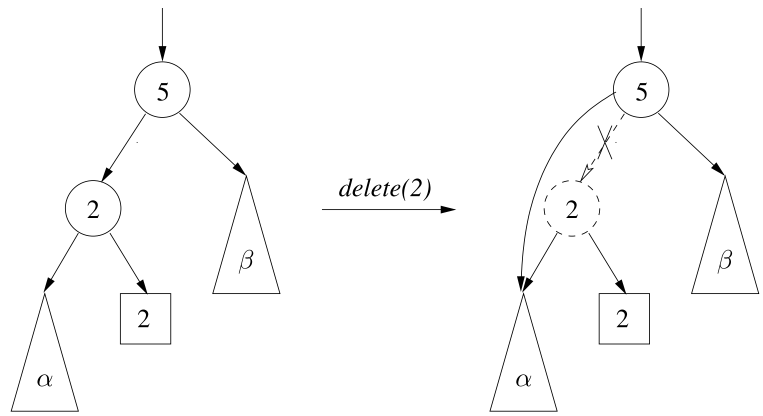


Figure 3.4: An example of a *delete* operation in a leaf-oriented BST.

key of the replaced leaf and the inserted key are in that subtree and their parent node contains the maximum of the two keys. After such an update, $insert(1)$ returns $true$.

A *delete* operation locates a leaf node that potentially contains the given key using the *search* routine as well. If such a node does not contain the given key, the *delete* operation returns $false$. Otherwise, the given key is detected, as in the example shown in Figure 3.4, and the child pointer of the leaf's grandparent is changed from the leaf's parent to the leaf's sibling and *true* is returned. This ensures that the deleted node is no longer *reachable* through the child pointers of the BST.

Some coordination between processes is needed to avoid problems when more than one process wants to update the same part of the tree concurrently. Partly inspired by Fomitchev and Ruppert's linked list implementation [26] and the cooperative technique of Barnes [27], the non-blocking BST algorithm uses a flagging system to indicate whether there is a process operating at a node. Intuitively, each internal node can be flagged and flags behave like a kind of lock. There are different types of flags used to represent different operations. When a node is flagged, only some particular steps can be applied to it to continue the operation that placed the flag. Other operations have to help this operation to complete before they can place their own flags. Every node has a field to indicate its current *state*. Initially, the *state* of a node is set to CLEAN. Before an *insert* or *delete* operation changes the child pointer of a node, the node's *state* must be set to IFLAG or DFLAG, respectively. After the child pointer is changed, the *state* of the node is set to CLEAN again. The *state* field of a node is flagged using a *CAS* step which succeeds only if the *state* of that node is CLEAN and has not changed since the operation read the node's child pointer. This guarantees that during the whole operation of a process, no other operations modify those flagged nodes.

However, these flag states are not sufficient for a *delete* operation. Figure 3.5 illustrates a problem when two simultaneous *delete* operations happen using flags only to "lock" the grandparents. In Figure 3.5(a), *delete*(5) and *delete*(1) occur concurrently. They set the states of the internal nodes with keys 6 and 4 to DFLAG at the same time before changing their child pointers. Initially the set contains keys $\{1, 2, 5, 7\}$. Then, both operations modify the child pointers of their flagged nodes

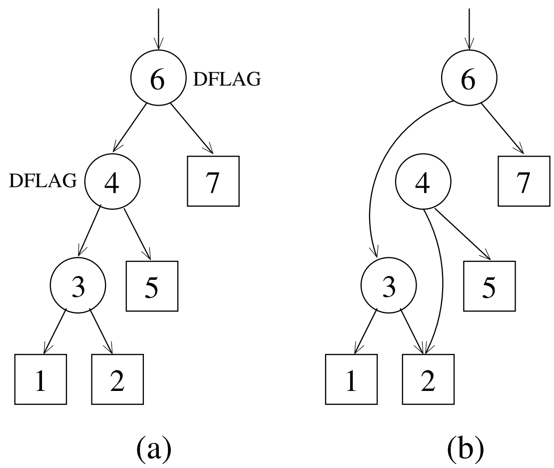


Figure 3.5: A problem caused by two *delete* operations if we only use the DFLAG state. (a). $delete(5)$ and $delete(1)$ are being executed and the nodes whose keys are 6 and 4 are set to DFLAG before changing their child pointers. (b). The BST after $delete(5)$ and $delete(1)$ were completed.

and the resulting subtree is shown in Figure 3.5(b), where the leaves contain keys $\{1, 2, 7\}$. This is because only leaf nodes containing 1, 2 and 7 are reachable from the *root* of the BST. However, according to the specification of the set (Definition 3.1), the BST should contain only $\{3, 7\}$ after those two *delete*s.

To solve this kind of problem, Ellen et al. introduce another MARK state. A *delete* operation must set the *state* of the leaf's parent to MARK before changing the grandparent's child pointer to remove the parent node from the tree. The *state* of a node can be set to MARK only if it is CLEAN, and once a node is marked, it remains so forever. Intuitively, the MARK state guarantees that a node cannot be set to MARK and DFLAG/IFLAG at the same time. Thus, when a *delete* operation removes a marked node from the BST, no operation can subsequently modify the marked node.

Because the flagging system intuitively behaves like locks, it may prevent progress. Figure 3.6 illustrates an example where no more operations can be done on the nodes whose keys are 3 and 4 due to the crash of $delete(1)$. The operation $delete(5)$ gets blocked because it attempts to MARK the node whose key is 4. However, that node is not in its CLEAN state. In order to guarantee the progress property of this algo-

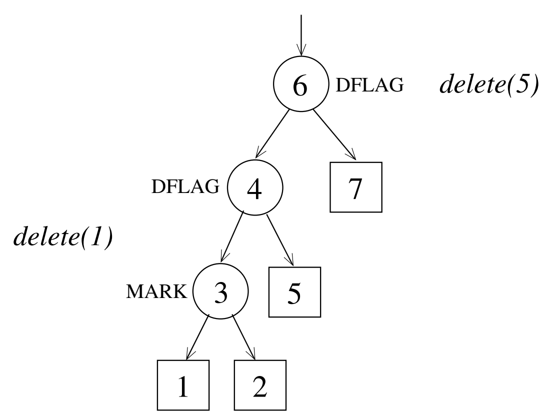


Figure 3.6: If $delete(1)$ dies, it blocks $delete(5)$.

rithm, Ellen et al. [3] used helping mechanisms in the *insert* and *delete* operations. Basically, besides setting the *states* of a node, every operation also stores some essential information about itself in that node. Thus, if an operation is blocked by an unfinished operation, it uses this information to try to help complete the unfinished one before restarting its own operation. To ensure that only one helper of an operation performs the required change to the tree, child pointers are also updated using *CAS* steps.

Figure 3.7 illustrates the big picture of how the *state* of a node changes during different steps of an *insert* or *delete* operation. Its right part, included in the blue box, describes steps of an *insert* operation. Refer to pseudocode in Figure 3.10 and 3.11. An *insert* operation tries to set a node's state from CLEAN to IFLAG by an iflag *CAS* (Line 31). After that, the *insert* operation changes its child pointer to a new subtree containing three nodes by an ichild *CAS* (Line 41) while the *state* of the node remains IFLAG. Subsequently, the operation changes the node with IFLAG *state* to a CLEAN node by an iunflag *CAS* (Line 43).

The rest of Figure 3.7 describes steps of a *delete* operation. A *delete* operation first flags a CLEAN grandparent node, changing its state to DFLAG by a dflag *CAS* (Line 54). Then, such a *delete* operation may continue or backtrack depending on whether it successfully marks the parent node by a mark *CAS* (Line 62) or not. If the mark *CAS* succeeds, the grandparent node's state remains unchanged and the parent node's state is changed from CLEAN to MARK. Subsequently, the *delete* operation changes the child pointer of the grandparent node and then sets it back to a CLEAN node by a dchild *CAS* (Line 85) and a dunflag *CAS* (Line 87), respectively. If the mark CAS fails, the *delete* operation backtracks and changes the grandparent node's state from DFLAG to CLEAN through a backtrack *CAS* (Line 80) and restarts the *delete* operation.

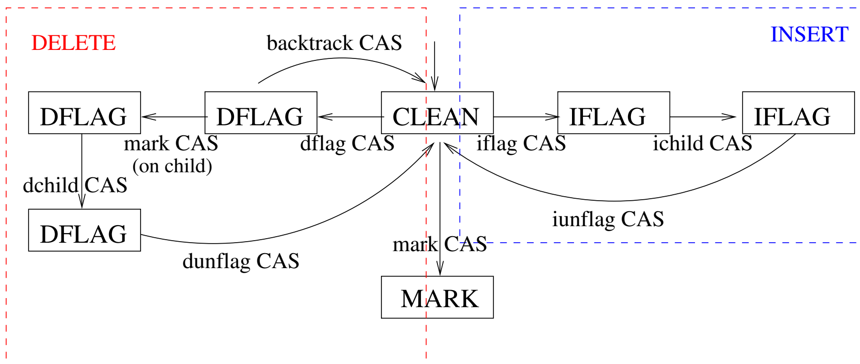


Figure 3.7: Main CAS steps and their effects of changing the states of nodes for a *delete* and *insert* operations.

3.1.2 Detailed Implementation

The non-blocking BST uses objects that support *read, write* and CAS operations. The key set U is totally ordered. To avoid special cases that would require changing the *root*, the tree is initialized as shown in Figure 3.8. We assume there are two

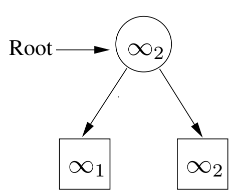


Figure 3.8: The initial state of the tree in the non-blocking BST algorithm.

special values ∞_1 and ∞_2, such that every value in U is less than ∞_1 and ∞_2, and $\infty_1 < \infty_2$. Hence, every insert or delete operation only modifies the left subtree of *root*. The types of objects we use to represent the data structure are defined in Figure 3.9. Internal nodes and leaf nodes are distinguished by the truth value of the

```
type Node{
    Key ∪ {∞1, ∞2} key
    Node left, right
    Info info
    Bool isinternal
}

type Info{
    {CLEAN, DFLAG, IFLAG, MARK} infotype
    Node gpn, pn, ln, nIntern,
    Info pinfo, dinfo
}
```

Figure 3.9: Data types defined in the non-blocking BST algorithm.

isinternal field of Node objects. For simplicity, both internal nodes and leaf nodes have *left* and *right* fields. However, for the leaf nodes, they all point to a special *NIL* Node. Every node has an *info* field, which points to an Info object. There are four types of Info objects, CLEAN, DFLAG, IFLAG and MARK, distinguished by the value of the *infotype* field. An Info object can also record essential information about an *insert* or *delete* operation. This information is stored in its *gpn, pn, ln, nIntern, pinfo* and *dinfo* fields when an Info object is created. A CLEAN Info object does not need to store any further information in those fields. An IFLAG Info object, which is created by an *insert* operation, usually stores the leaf node to be replaced in its ln field, the parent of that leaf node in pn, and the newly created internal node in $nIntern$. A DFLAG Info object, which is created by a *delete* operation, stores the leaf node to be removed, the parent of the leaf node and the grandparent of the leaf node in ln, pn and gpn, respectively. It also stores an Info object that was read from the parent in $pinfo$. (This is used by other processes helping the *delete* as the old value for the mark *CAS*.) A MARK Info object, which is also created by a *delete* operation after the creation of a DFLAG Info object, just has a pointer to the DFLAG Info object created by the deletion.

The detailed implementations of the non-blocking algorithms are shown in Figure 3.10 and Figure 3.11, where comments are preceded by ▷ . Basically, all three operations call the sub-routine $Search(k)$ to traverse nodes until reaching a leaf. The $Search(k)$ routine takes a key k as its input parameter and returns five objects. At Line 2, the search starts from the *root*. The search goes down to the left or right child depending on whether the key field of the current internal node is less or greater than the given key k. It stops when it hits a leaf node (Line 4). During the while loop, it stores the last three visited nodes as gpn, pn and ln (grandparent, parent and leaf node). It also stores the $info$ field of gpn and pn. A $Find(k)$ operation calls $Search(k)$ and gets the returned leaf node. If the *key* field of the leaf node is equal to k, it returns *true*, otherwise it returns *false*.

Definition 3.3. The *sequence of visited nodes* by an invocation of *search* is the

```
Search(Key k) : ⟨Node, Node, Node, Info, Info⟩ {
    ⊳ Used by Insert, Delete and Find to traverse a branch of the BST; satisfies following postconditions:
    ⊳ (1) ln points to a Leaf node and pn points to an Internal node
    ⊳ (2) Either pn.left has contained ln (if k < pn.key) or pn.right has contained ln (if k ≥ pn.key)
    ⊳ (3) pn.info has contained pinfo
    ⊳ (4) if ln.key ≠ ∞1, then the following three statements hold:
    ⊳    (4a) gpn points to an Internal node
    ⊳    (4b) either gpn.left has contained pn (if k < gpn.key) or gpn..right has contained pn (if k ≥ gpn.key)
    ⊳    (4c) gpn.info has contained gpinfo
    Node gpn, pn, ln := Root
    Info gpinfo, pinfo                                       ⊳ Each stores a copy of an info field

    while ln points to an internal node {
        gpn := pn                                            ⊳ Remember parent of pn
        pn := ln                                             ⊳ Remember parent of ln
        gpinfo := pinfo                                      ⊳ Remember info field of gpn
        pinfo := pn.info                                     ⊳ Remember info field of pn
        if k < ln.key then ln := pn.left else ln := pn.right ⊳ Move down to appropriate child
    }
    return ⟨gpn, pn, ln, pinfo, gpinfo⟩
}

Find(Key k) : boolean {
    Node ln

    ⟨−, −, ln, −, −⟩ := Search(k)
    if ln.key = k then return true
    else return false
}

Insert(Key k) : boolean {
    Node ln, pn, nIntern, nSib
    Node nNode := a new leaf node whose key field is k
    Info pinfo, result, op

    while TRUE {
        ⟨−, pn, ln, pinfo, −⟩ := Search(k)
        if ln.key = k then return false                      ⊳ Cannot insert duplicate key
        if pinfo.infotype ≠ CLEAN then Help(pinfo)           ⊳ Help the other operation
        else {
            nSib := a new leaf whose key is ln.key
            nIntern := a new internal node with key field max(k, ln.key),
                info field ⟨CLEAN, ⊥, ⊥⟩, and with two child fields equal to nNode and nSib
                (the one with the smaller key is the left child)
            op := a new Info object containing ⟨IFLAG, pn, ln, nIntern⟩
            result := CAS(pn.info, pinfo, op)                ⊳ iflag CAS
            if result = pinfo then {                         ⊳ The iflag CAS was successful
                HelpInsert(op)                               ⊳ Finish the insertion
                return TRUE
            }
            else Help(result)          ⊳ The iflag CAS failed; help the operation that caused failure
        }
    }
}

HelpInsert(Info op) {
    ⊳ Precondition: op is to an IFLAG Info object (i.e., it is not ⊥)
    CAS-Child(op.pn, op.ln, op.nIntern)                      ⊳ ichild CAS
    clean := a new CLEAN Info object
    CAS(op.pn.info, op, clean)                               ⊳ iunflag CAS
}
```

Figure 3.10: Pseudocode for Search, Find and Insert [3].

```
Delete(Key k) : boolean {
    Node gpn, pn, ln; Info pinfo, gpinfo, result, op;

    while TRUE {
        ⟨gpn, pn, ln, pinfo, gpinfo⟩ := Search(k)
        if ln.key ≠ k then return false                          ▷ Key k is not in the tree
        if gpinfo.infotype ≠ CLEAN then Help(gpinfo)
        else if pinfo.infotype ≠ CLEAN then Help(pinfo)
        else {                                                          ▷ Try to flag gpn
            op := a new DFLAG Info object containing ⟨gpn, pn, ln, pinfo⟩
            result := CAS(gpn.info, gpinfo, op)                         ▷ dflag CAS
            if result = gpinfo then {                             ▷ CAS successful
                if HelpDelete(op) then return true                ▷ Either finish deletion or unflag
            }
            else Help(result)         ▷ The dflag CAS failed; help the operation that caused the failure
        } } }

HelpDelete(Info op) : boolean {
    ▷ Precondition: op points to a DFLAG Info object (i.e., it is not ⊥)
    Info result, result2, op2, op3, clean

    op2 := a new MARK Info object ⟨MARK, dinfo := op⟩
    result := CAS(op.pn.info, op.pinfo, op2)                            ▷ mark CAS
    if result = op.pinfo or [ result.infotype = MARK, result.dinfo = op ] then {
        ▷ op.pn is successfully marked
        HelpMarked(op)                                            ▷ Complete the deletion
        return true                                               ▷ Tell Delete routine it is done
    }
    else {                                                        ▷ The mark CAS failed
        if result.infotype = IFLAG then HelpInsert(result)        ▷ op.pn is an IFLAG node
        if result.infotype = MARK then HelpMarked(result.dinfo)   ▷ op.pn is a MARK node
        if result.infotype = DFLAG then {                         ▷ op.pn is a DFLAG node
            op3 := a new MARK Info object ⟨MARK, dinfo := result⟩
            ▷ Non-recursively help the DFLAG node
            result2 = CAS(result.pn.info, result.pinfo, op3)
            if result2 = result.pinfo or [ result2.infotype = MARK, result2.dinfo = result ]
            then HelpMarked(result)
            else {
                clean := a new CLEAN Info object                  ▷ The non-recursive mark help fails
                CAS(result.gpn.info, result, clean)               ▷ Help op.pn backtrack
            }
        }
        clean := a new CLEAN Info object
        CAS(op.gpn.info, op, clean)                               ▷ backtrack CAS
        return false                                              ▷ Tell Delete routine to try again
    } }

HelpMarked(Info op) {
    ▷ Precondition: op points to a DFLAG Info object (i.e., it is not ⊥)
    Node other; Info clean;

    ▷ Set other to point to the sibling of the node to which op.ln points
    if op.pn.right = op.ln then other := op.pn.left else other := op.pn.right
    ▷ Splice the node to which op.pn points out of the tree, replacing it by other
    CAS-Child(op.gpn, op.pn, other)                                     ▷ dchild CAS
    clean := a new CLEAN Info object
    CAS(op.gpn.info, op, clean)                                         ▷ dunflag CAS
}
```

Figure 3.11: Pseudocode for Delete and some auxiliary routines [3].

```
90  Help(Info u) {
        ▷ General-purpose helping routine
        ▷ Precondition: u has been stored in the info field of some internal node
91      if u.infotype = IFLAG then HelpInsert(u)
92      else if u.infotype = MARK then HelpMarked(u)
93      else if u.infotype = DFLAG then HelpDelete(u)
94  }
95  CAS-Child(Node parent, Node old, Node new) {
        ▷ Precondition: parent points to an Internal node and new points to a Node (i.e., neither is ⊥)
        ▷ This routine tries to change one of the child fields of the node that parent points to from old to new.
96      if new.key < parent.key then
97          CAS(parent.left, old, new)
98      else
99          CAS(parent.right, old, new)
100 }
```

Figure 3.12: Pseudocode for Delete and some auxiliary routines [3].

sequence of nodes $\langle n_0, n_1, n_2, \cdots, n_m \rangle$ that ln points to. More specifically, n_0 is *root* since ln is first set to *root* on Line 2. For $1 \leq i \leq m$, n_i is the node that ln points to immediately after ln gets updated by the ith iteration of Line 8.

An $Insert(k)$ operation first creates a new leaf node containing k at Line 21. Then, it tries to insert this leaf until it succeeds. In a single iteration of the loop, if the leaf node returned by a $Search(k)$ sub-routine does not contain k, and no other operation was changing pn, it creates a subtree containing three nodes (Line 28-29). After that, the operation creates an Info object that stores the information about the operation (Line 30) and tries to flag pn. If the flagging succeeds, the operation changes a child pointer of pn from ln to the newly created subtree and unflags the node with IFLAG state to CLEAN (Line 42-43). If the flagging was blocked by another unfinished operation, the *search* tries to help the other operation and then starts its own work again (Line 36).

A $Delete(k)$ operation returns $false$ if the $Search(k)$ returns a leaf node which does not contain k. Otherwise, the $Delete(k)$ operation tries to remove the leaf returned by the $Search(k)$ from the BST. It consists of three main steps: flag the grandparent node gpn using a dflag CAS (Line 53-54), mark the parent node pn using a mark CAS (Line 61-62), and change the child pointer using a dchild CAS (Line 87). If the dflag CAS on gpn is blocked by another unfinished operation, the delete helps the unfinished operation (Line 58). After setting the *state* of gpn to DFLAG, it attempts a mark CAS on pn. If this mark CAS is blocked by another unfinished operation, the delete helps the unfinished one (Line 67-78) and backtracks (*i.e.*, performs Line 79-80 to set gpn's state to CLEAN) and starts a new iteration of $Delete(k)$. Otherwise, the mark CAS succeeds and the *delete* operation continues by performing a dchild CAS (Line 85) to change the child pointer and then resetting the state of gpn to CLEAN using a dunflag CAS (Line 87).

3.2 A Simplified Algorithm

To make our verification of the proof of correctness easier, we introduce a simplified version of the non-blocking BST algorithms without helping mechanisms and prove this new version correct in PVS using simulations. Once this proof is complete, we believe it will be possible to extend it to prove the correctness of the original algorithm. The ideas behind the simplified algorithm are the same as the original one, except that if an operation is blocked by other unfinished operations, it tries again and until the unfinished one gets finished. This technique is called busy waiting, and does not guarantee the progress property. The pseudocode for the simplified algorithms is shown in Figure 3.13 and 3.14.

We use Si, Fi, Ii and Di to represent the ith Line in the Search, Find, Insert and Delete pseudocode in Figure 3.13 and 3.14, respectively. In the simplified version of the BST algorithm, we have made a few changes. In the original paper, a bit in the word of the node's pointer to an Info object represents the type of Info object. But we use the $infotype$ (CLEAN/IFLAG/DFLAG/MARK) field inside the Info object to distinguish them. This makes it more clear and straightforward when we implement the algorithm. As a consequence, we always create new CLEAN objects to avoid the ABA problem.

The main steps of Find, Insert and Delete operations are the same as in the original algorithms. The subroutine $Search(k)$ remains the same as before and is used by all $Find(k)$, $Insert(k)$ and $Delete(k)$ operations. The $Find(k)$ operation is exactly the same. The $Insert(k)$ operation inserts a node containing k (created at I1) if there is no such leaf node containing k found by $Search(k)$. First, it calls the subroutine $Search(k)$ to determine if there is a leaf node that potentially contains k (I2). If such a leaf does not exist, the operation attempts to insert the key into the BST. From I6 to I9, a new subtree containing three nodes is created. The operation then attempts to set the state of pn to IFLAG by a iflag CAS (I10-I11). If this iflag CAS is blocked by an other unfinished operation, it loops and tries again. Otherwise, after a successful iflag CAS, it changes the child pointer of pn from ln to the newly created subtree by an ichild CAS (I13-I15). The operation changes the state of pn to CLEAN by an iunflag CAS (I16-I17).

A $Delete(k)$ operation searches the BST to check if there is a node potentially containing k (D1). It returns $false$ if the leaf node returned by $Search(k)$ does not contain k. Otherwise, the operation sets gpn's state to DFLAG by a dflag CAS (D6-D7). If the dflag CAS is blocked by some other unfinished operations, the current $Delete(k)$ loops and attempts again. After a successful dflag CAS (D8), the *delete* operation tries to set the state of pn to MARK by a mark CAS (D9-D10). If the mark CAS is blocked by some other unfinished operation, it backtracks (D20) and starts $Delete(k)$ again. If the mark CAS succeeds (D12), the operation then changes the child pointer of gpn from pn to the sibling of ln using a dchild CAS (D17-D18), thereby deleting ln from the BST. After the dchild CAS, a dunflag CAS (D19) resets the state of gpn to CLEAN.

```
Search(Key k) : <Node, Node, Node, Info, Info>
ln ← Root
while ln is not a leaf {
      gpn ← pn
      pn ← ln
      gpinfo ← pinfo
      pinfo ← pn.info
      if k < ln.key
            ln ← pn.left
      else ln ← pn.right
}
```

```
Find(Key k) : Node
<−, −, ln, −, −> ← SEARCH(k)
if ln.key = k
      return true
else return false
```

```
Insert(Key k) : boolean
nNode ← newNode(key← k, isleaf← true, isinternal← false)
while TRUE {
      <−, pn, ln, pinfo, −> ← SEARCH(k)
      lnk ← ln.key
      if lnk = k
            return false
      if pinfo.infotype = CLEAN {
            nSib ← newSib(key← lnk, isleaf← true, isinternal← false)
            if k > lnk
                  nIntern ← newIntern(key← k, left← nSib, right← nNode,
                                      isleaf← false, isinternal← true)
            else
                  nIntern ← newIntern(key ← lnk, left← nNode, right← nSib,
                                      isleaf← false, isinternal← true)
            op ← newIInfo(IFLAG, pn, ln, nIntern)
            result ← CAS(pn.info, pinfo, op)
            if result = pinfo {
                  if op.nIntern.key < op.pn.key
                        CAS(op.pn.left, op.ln, op.nIntern)
                  else CAS(op.pn.right, op.ln, op.nIntern)
                  clean ← newCInfo(CLEAN, −, −, −)
                  CAS(op.pn.info, op, clean)
                  return true
            }
      }
}
```

Figure 3.13: Pseudocode for Search and Find operations.

```
Delete(Key k) : boolean
    while TRUE {
        <gpn, pn, ln, pinfo, gpinfo> ← SEARCH(k)
        lnk ← ln.key
        if lnk ≠ k
            return false
        if gpinfo.infotype = CLEAN {
            if pinfo.infotype = CLEAN {
                op1 ← newDInfo(DFLAG, gpn, pn, ln, pinfo)
                result ← CAS(gpn.info, gpinfo, op1)
                if result = gpinfo {
                    op2 ← newMInfo(MARK, dinfo ← op1)
                    result ← CAS(op1.pn.info, op1.pinfo, op2)
                    clean ← newCInfo(CLEAN, −, −, −)
                    if result = op1.pinfo
                        if op1.pn.right = op1.ln
                            other ← op1.pn.left
                        else other ← op1.pn.right
                        if other.key < op1.gpn.key
                            CAS(op1.gpn.left, op1.pn, other)
                        else CAS(op1.gpn.right, op1.pn, other)
                        CAS(op1.gpn.info, op1, clean)
                        return true }
                    else {
                        CAS(op1.gpn.info, op1, clean)
        } } } }
    }
```

Figure 3.14: Pseudocode for Delete operations.

4 Modelling the Algorithms

In order to prove the correctness of the simplified BST algorithm using PVS, we model the implementation and the specification as automata which are called the concrete automaton and canonical automaton, respectively. To make the proof easier, we introduce an intermediate automaton and use a hybrid forward and backward simulation to prove correctness. We show that the concrete automaton implements the intermediate one via a forward simulation and the intermediate automaton implements the canonical one via a backward simulation.

4.1 The Canonical Automaton

The canonical automaton models the abstract specifications of the $\mathcal{SET}$ data type defined in Definition 3.1. By using the method introduced in Section 2.3.2, we can build the canonical automaton easily.

As mentioned in Section 3.1.2, let U be a totally ordered set and $UPlus = U \cup \{\infty_1, \infty_2\}$ such that every value in U is less than ∞_1 and ∞_2 and $\infty_1 < \infty_2$. Intuitively, U contains all possible keys that can be inserted into the data structure.

Let $PROC$ be a finite set of processes. Let $Pcval$ be the set of all possible values for the program counter of a process. More precisely, we define $Pcval$ as follows.

$$\begin{aligned} Pcval = \Big\{ & idle, \\ & pcDoFind(k), pcFindResp(true), pcFindResp(false), \\ & pcDoInsert(k), pcInsertResp(true), pcInsertResp(false), \\ & pcDoDelete(k), pcDeleteResp(true), pcDeleteResp(false) \mid k \in U \Big\}. \end{aligned}$$

The *state* of the canonical automaton $AbsAut$ is a pair: $(keys, pc)$, where $keys \subseteq U$ and $pc : PROC \longrightarrow Pcval$. The initial state *start* in the canonical automaton $AbsAut$ has $start.keys = \emptyset$ and $start.pc(p) = pc\_idle$ for all $p \in PROC$. In PVS, we model a state of the $AbsAut$ as follows:

$$\begin{aligned} \text{state : TYPE} = \{ \ & \text{keys} : setof[U], \\ & \text{pc} : [PROC \rightarrow Pcval] \ \}. \end{aligned}$$

$PROC$ is modelled as subset of the natural numbers from 0 to some $n \geq 1$ in PVS. We use $setof[U]$ to model a set whose elements are all in U. Thus, state.keys records the set of keys the BST currently contains, and state.pc records the program counter of each process.

Figure 4.1 shows all *external* and *internal* actions for $AbsAut$. For each kind of operation, two different *internal* actions are used to capture the linearization points

$in(AbsAut) = \{findInv_p(k), insertInv_p(k), deleteInv_p(k) \mid k \in U, p \in PROC\}$

$out(AbsAut) = \{findResp_p(r), insertResp_p(r), deleteResp_p(r) \mid r \in boolean, p \in PROC\}$

$int(AbsAut) = \{doFindT_p(k), doFindF_p(k), doInsertT_p(k), doInsertF_p(k), doDeleteT_p(k), doDeleteF_p(k) \mid k \in U, p \in PROC\}$

Figure 4.1: *Actions* of the canonical automaton *AbsAut* for a $\mathcal{SET}$ data type.

of operations that return *true* or *false*. All transitions for the *AbsAut* are defined in Table 4.1. To make the description similar to our formalization in PVS, we use *keys*.add(k) or *keys*.remove(k) to represent adding or removing an element k from a set *keys*.

4.2 The Concrete Automaton

The concrete automaton *ConcAut* is used to represent the implementation. This automaton models the pseudocode we described in Figure 3.13 and 3.14. More details of modelling the *ConcAut* in PVS can be found in our PVS scripts. We only discuss some key parts of the modelling here.

A state of *ConcAut* contains four parts: program counters, local variables, shared objects in shared memory and auxiliary variables. The program counter of a process records which line of code the process will next execute. We define a set *Pcval*

Action	**Precondition**	**Effect**
$findInv(k,p)$	$s.\mathrm{pc}(p) = idle$	$s.\mathrm{pc}(p) \leftarrow pcDoFind(k)$
$doFindT(k,p)$	$s.\mathrm{pc}(p) = pcDoFind(k)$ $k \in s.keys$	$s.\mathrm{pc}(p) \leftarrow pcFindResp(true)$
$doFindF(k,p)$	$s.\mathrm{pc}(p) = pcDoFind(k)$ $k \notin s.keys$	$s.\mathrm{pc}(p) \leftarrow pcFindResp(false)$
$findResp(r,p)$	$s.\mathrm{pc}(p) = pcFindResp(r)$	$s.\mathrm{pc}(p) \leftarrow idle$
$insertInv(k,p)$	$s.\mathrm{pc}(p) = idle$	$s.\mathrm{pc}(p) \leftarrow pcDoInsert(k)$
$doInsertT(k,p)$	$s.\mathrm{pc}(p) = pcDoInsert(k)$ $k \notin s.keys$	$s.\mathrm{pc}(p) \leftarrow pcInsertResp(true)$ $s.keys.\mathrm{add}(k)$
$doInsertF(k,p)$	$s.\mathrm{pc}(p) = pcDoInsert(k)$ $k \in s.keys$	$s.\mathrm{pc}(p) \leftarrow pcInsertResp(false)$
$insertResp(r,p)$	$s.\mathrm{pc}(p) = pcInsertResp(r)$	$s.\mathrm{pc}(p) \leftarrow idle$
$deleteInv(k,p)$	$s.\mathrm{pc}(p) = idle$	$s.\mathrm{pc}(p) \leftarrow pcDoDelete(k)$
$doInsertT(k,p)$	$s.\mathrm{pc}(p) = pcDoDelete(k)$ $k \in s.keys$	$s.\mathrm{pc}(p) \leftarrow pcDeleteResp(true)$ $s.keys.\mathrm{remove}(k)$
$doDeleteF(k,p)$	$s.\mathrm{pc}(p) = pcDoDelete(k)$ $k \notin s.keys$	$s.\mathrm{pc}(p) \leftarrow pcDeleteResp(false)$
$deleteResp(r,p)$	$s.\mathrm{pc}(p) = pcDeleteResp(r)$	$s.\mathrm{pc}(p) \leftarrow idle$

Table 4.1: Transitions of the canonical automaton $AbsAut$, where s is a variable of TYPE $state$, k is an element of U and P is an element of $PROC$.

of possible values for a process's program counter. Intuitively, each line of the pseudocode is modelled as an element in $Pcval$.

$$
\begin{aligned}
Pcval = \Big\{ & idle, \\
& pcSearch1, pcSearch2, \cdots, pcSearch9, \\
& pcFind1, pcFind2, pcFindResp(r), \\
& pcInsert1, pcInsert2, \cdots, pcInsert17, pcInsertResp(r), \\
& pcDelete1, pcDelete2, \cdots, pcDelete20, pcDeleteResp(r) \mid r \in boolean \Big\}.
\end{aligned}
$$

Then, the component *pc* of the state of *ConcAut* is a function $pc : PROC \longrightarrow Pcval$.

The way to model shared objects in *ConcAut* is a bit tricky. Node and Info objects which are defined in Figure 3.9 are modelled as two abstract types in PVS called Node and Info. Their fields, such as the child pointers of a node, the key field of a node or the leaf field of an Info object are modelled as functions from Node (Info) to the desired type. For clarity, the name of each field has a "f" as suffix. Thus, shared variables are modelled by the functions described in Table 4.2.

One can easily construct the types in Table 4.2 from Figure 3.9. The Flag type is defined by: Flag TYPE = {CLEAN, DFLAG, IFLAG, MARK}, as described in Section 3.1.2.

In order to record the local information of each process, each local variable is modelled by a component of the state in *ConcAut*. Because these variables are local, they are modelled as functions from processes to the appropriate type, as listed in Table 4.3.

Node object			Info object	
shared variable	**function**		**shared variable**	**function**
keyf	Node $\longrightarrow$ UPlus		infotypef	Info $\longrightarrow$ Flag
leftf	Node $\longrightarrow$ Node		gpnf	Info $\longrightarrow$ Node
rightf	Node $\longrightarrow$ Node		pnf	Info $\longrightarrow$ Node
infof	Node $\longrightarrow$ Info		lnf	Info $\longrightarrow$ Node
isinternf	Node $\longrightarrow$ boolean		nInternf	Info $\longrightarrow$ Node
			pinfof	Info $\longrightarrow$ Info
			dinfo	Info $\longrightarrow$ Info

Table 4.2: Representing fields of shared objects in the state of *ConcAut*.

local variable	**function**		**local variable**	**function**
ret_addr	PROC $\longrightarrow$ pc_return		k	PROC $\longrightarrow$ U
lnk	PROC $\longrightarrow$ UPlus		pn	PROC $\longrightarrow$ Node
gpn	PROC $\longrightarrow$ Node		ln	PROC $\longrightarrow$ Node
other	PROC $\longrightarrow$ Node		result	PROC $\longrightarrow$ Info
gpinfo	PROC $\longrightarrow$ Info		pinfo	PROC $\longrightarrow$ Info
op	PROC $\longrightarrow$ Info		op1	PROC $\longrightarrow$ Info
op2	PROC $\longrightarrow$ Info		clean	PROC $\longrightarrow$ Info
nSib	PROC $\longrightarrow$ Node		nIntern	PROC $\longrightarrow$ Node

Table 4.3: Local variables of a state in *ConcAut*.

All local variables in Table 4.3 are straightforward to obtain from the simplified algorithm, except for ret_addr. This local variable is used when the *search* subroutine is invoked and it records where to continue from if the subroutine completes. Hence, $pc\_return = \{pcFind2, pcInsert3, pcDelete2\} \subseteq Pcval$. The states of *ConcAut* also include auxiliary variables: $aux\_keys \subseteq U$, *aux_seen_in*, *aux_seen_out* : $PROC \longrightarrow boolean$. They do not model anything in the pseudocode, but are used to simplify our proofs. They are discussed in Section 4.4.

The initial state of $ConcAut$ is defined as follows:

* Most local variables are initialized to NIL, except that for all p : pc(p)=$idle$, lnk(p)=∞_2.

* The value of some fields of shared objects, namely the keyf and isinternf fields of a Node object and the infotypef field of an Info object, is not specified. Their initial value are irrelevant to some lemmas we need to prove later.

* Most fields of shared objects are initialized to NIL, except three shared Node objects listed in Table 4.4 and three Info objects listed in Table 4.5.

* The initial values of auxiliary variables are: $aux\_keys = \emptyset$, $aux\_seen\_in(p) = false$, $aux\_seen\_out(p) = false$ for all $p \in PROC$. (More details are discussed in Section 4.4.)

There are three allocated Nodes: $root$ and its two children ($nInf1$ and $nInf2$) in an initial state, as well as the Info objects that belong to them ($CL1$, $CL2$ and $CL3$).

As discussed in Section 2.4.1, the idea of building the $ConcAut$ is straightforward: each line of the code, which contains at most one shared memory access, is modelled by a single internal action except for an *if* statement, a test of the exit condition of a *while* loop or a CAS operation. Each of those three types of lines are modelled by

field f	**value of** $f(root)$	**value of** $f(nInf1)$	**value of** $f(nInf2)$
keyf	∞_2	∞_1	∞_2
leftf	nInf1	NIL	NIL
rightf	nInf2	NIL	NIL
infof	CL1	CL2	CL3
isinternf	$true$	$false$	$false$

Table 4.4: Initial state of *root* and its two children.

field f	**value of** $f(CL1)$	**value of** $f(CL2)$	**value of** $f(CL3)$
infotypef	CLEAN	CLEAN	CLEAN
gpnf	NIL	NIL	NIL
pnf	NIL	NIL	NIL
lnf	NIL	NIL	NIL
nInternf	NIL	NIL	NIL
pinfof	NIL	NIL	NIL
dinfo	NIL	NIL	NIL

Table 4.5: Initial state of Info objects belong to *root* and its two children.

two actions: a successful one and a failed one. However, there are some actions in our concrete automaton consisting of several shared memory access. That is allowed, because only one of them accesses a changeable field, and the others are reading from unchangeable fields. Hence, it does not matter if we collapse steps that read from unchangeable fields into one action. For instance, when an Info object is created, the values of its fields remain unchanged. Hence, each of Lines I14, I15, I17, D10, D17, D18, D19 and D20 can be regarded as an atomic action in our concrete automaton, thereby simplifying our model of the concrete automaton. In addition to the internal actions, for each kind of operation, we define two external actions: an invocation and a response action. In the same way we modelled the implementation of the *fetch-and-increment* object in Table 2.3, we model the BST algorithm as follows. Most of steps in the pseudocode can be trivially translated into a transition in $ConcAut$, except for a few cases. All three operations ($find, insert$ and $delete$) invoke the $search$ subroutine. When the invocation of process p occurs, we set ret_addr(p) to the appropriate return address, while changing pc(p) to $pcSearch1$. Another interesting case is to model an allocation step in the pseudocode. We introduce two new variables $allocatedNode$ and $allocatedInfo$ in the state of $ConcAut$, which maintain a set of used Nodes and Info objects, respectively. Hence, whenever an allocation step for a Node is performed by a process p, we pick a node that is not in $allocatedNode$ and return the node to p, add the node to $allocatedNode$, and then assign appropriate values to its fields. This can be done by assuming an axiom that there are always infinitely many unallocated nodes to pick. Allocation of an Info object is done in the same way.

Table 4.6 shows some examples of modelling lines of the pseudocode. Whenever an invocation is performed at state s by a process p, the key k is saved into the local variable $s.k(p)$. The function newNode shown in Figure 4.2 behaves exactly as we discussed above. More precisely, when p creates a new $nNode$ at Line I1, it picks an unused Node object n and Info object x and adds them into *allocatedNode* and *allocatedInfo* set, sets the type of x to CLEAN, points $infof(n)$ to x, assigns the node a key value and sets $isinternf(n)$ to $false$. If p creates a MARK Info object through the newMInfo function that has three parameters, p picks an unused Info object x and adds it into the *allocatedInfo* set, sets $infotypef(x)$ to MARK, points $dinfof(x)$ to an Info object and returns this newly allocated MARK Info object to $op2(p)$.

newNode($c : state, p : PROC, k : U$) :

LET n = getNode(c), x = getInfo(c).

c.allocatedNode.add(n)

c.allocatedInfo.add(x)

c.infotypef(x) $\leftarrow$ CLEAN

c.infof(n) $\leftarrow x$

c.isinternf(n) $\leftarrow false$

c.keyf(n) $\leftarrow k$

c.nNode(p) $\leftarrow n$

newMInfo($c : state, p : PROC, dinfo : Info$) :

LET x = getInfo(c).

c.allocatedInfo.add(x)

c.infotypef(x) $\leftarrow$ MARK

c.dinfof(x) $\leftarrow dinfo$

c.op2(p) $\leftarrow x$

Figure 4.2: Definition of the newNode and newMInfo functions. Function getNode(c) picks a new Node object that is not in c.*allcoatedNode*. Symmetrically for getInfo(c).

4.3 An Intermediate Automaton and Backward Simulation

In the concurrent BST algorithm [3], some operations, such as a $find(k)$ operation by a process p that returns *true*, may not actually "take effect" at the time when p determines at Line F2 that $find(k)$ should return *true* by comparing k to the key of the leaf reached by the *search*. Consider the example shown in Figure 4.3 (a), which involves three processes. Processes q_0 and q_1 insert and then delete key 3, while p is concurrently executing $find(3)$. The other diagrams in Figure 4.3 describe the shape of the BST, as a result of those operations. Note that, when p successfully finds the desired key in the leaf, that leaf is no longer in the BST. Namely, when process p executes Line F2 and decides to return *true* for the $find(3)$ operation, that key is not in the BST. So, Line F2 can not be used as a linearization point of the Find operation. Similarly, I4 and D3 are not the linearization points for some failed Insert and Delete operations, respectively. For those operations, it is not obvious how to define the linearization points explicitly. The proofs by Ellen et al. [3] show that there exists a time during the *search* when the leaf eventually reached is in the BST. However, at that time, it is not known where the search will eventually end up. So, it is difficult to recognize the linearization point of a search when it happens, without knowing the future actions of that search. Hence, it is difficult to come up with a forward simulation from *ConcAut* to *AbsAut* such that some actions in *ConcAut* are directly mapped to the internal actions of *AbsAut* that represent linearization points. Therefore, we use a hybrid forward and backward simulation by building an intermediate automaton $IntAut$, such that this intermediate automaton simulates the canonical one via a backward simulation and the concrete automaton simulates the intermediate one via a forward simulation.

Since a proof using backward simulation is usually conceptually harder than a proof using a forward simulation, we chose to make $IntAut$ as similar to $AbsAut$ as possible to make the backward simulation proof easier [16]. Thus, most of the components of the intermediate automaton will be similar to those in $AbsAut$. Each process has two additional local boolean variables $seen\_in(p)$ and $seen\_out(p)$, which are inspired by the work of Colvin et al. [16]. If p is performing $find(k)$ or $insert(k)$, the variable $seen\_in(p)$ is set to be $true$ if k is in the key set of $IntAut$ either at the invocation p's operation, or k was not in the key set at the beginning but during p's operation some other $insert(k)$ operation by q successfully inserts k into the key set of $IntAut$. Symmetrically, $seen\_out(p)$ is set to be $true$ if k is not in the key set at the invocation of p's $find(k)$ or $delete(k)$, or some $delete(k)$ by another process successfully deletes k from the key set of $IntAut$ during p's $find(k)$ or $delete(k)$. Intuitively, these two variables record whether the desired key has been in the BST at any time since the beginning of the present operation, with the aim of helping process p to determine the return value of its operation. For instance, if $seen\_in(p)$ is $true$ when p is performing a $find(k)$ operation, it means that the key k has been

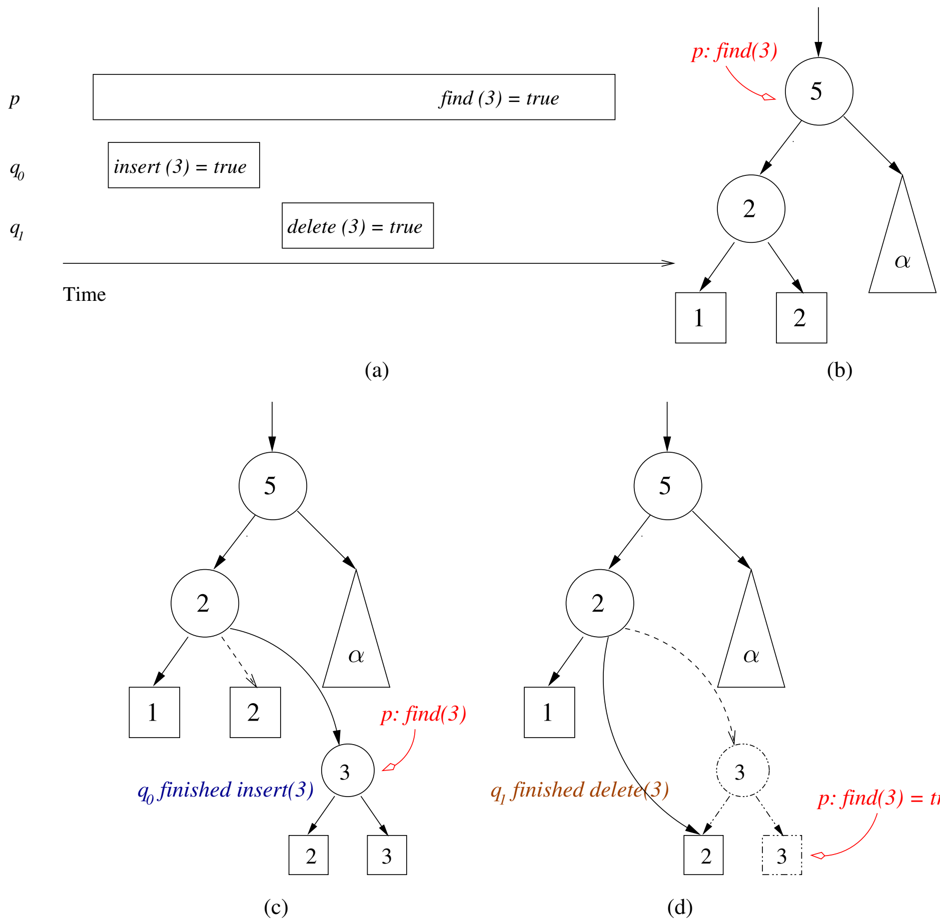


Figure 4.3: (a) Interleavings of Proc p, q_0 and q_1 which execute $find(3)$, $insert(3)$ and $delete(3)$, respectively (all three operations succeed). (b),(c),(d) illustrate how the three operations modify the binary search tree. (b) Proc p invoked $find(3)$ and it has set its local variable ln to the internal node with key 5. (c) Proc q_0 runs very quickly and successfully inserts key 3. Subsequently, p continues to search for key 3 and has set its local variable ln to (arrives at) internal node with key 3 on Line S9. (d) Proc q_1 executes a complete deletion of key 3, but after that p is still able to get to the external node with key 3 and subsequently return $true$.

in the key set at some time since the invocation of $find(k)$. This, indeed, enables the *find* operation to return *true* even if when executing F2, the key k is actually not in the set any more.

A *state* of the $IntAut$ is the tuple

$$(keys, pc, seen\_in, seen\_out), \text{ where } \begin{cases} keys \subseteq U, \\ pc : PROC \longrightarrow Pcval, \\ seen\_in : PROC \longrightarrow boolean, \\ seen\_out : PROC \longrightarrow boolean, \end{cases}$$

and U, $PROC$ and $Pcval$ are defined as in the definition of $AbsAut$. The possible values for $seen\_in$ and $seen\_out$ are $true$ and $false$. The initial states $start$ and actions for the $IntAut$ are shown in Figure 4.4.

We define the *states* of $IntAut$ in PVS as:

$$\begin{aligned} \text{state : TYPE} = \{\ & \text{pc : [PROC} \rightarrow \textit{Pcval}], \\ & \text{keys : setof}[U], \\ & \text{seen\_in : [PROC} \rightarrow \textit{bool}], \\ & \text{seen\_out : [PROC} \rightarrow \textit{bool}]\ \}\ . \end{aligned}$$

The actions for $IntAut$ are shown in Table 4.7. Intuitively, $seen\_in(p)$ and $seen\_out(p)$ are initialized during the invocation of each $find, insert$ and $delete$ operation of process p. The response of a $find(k)$ operation of process p now depends on the value of $seen\_in(p)$ and $seen\_out(p)$. An $insert_p(k)$ can decide to

$start = \{s \mid s.keys = \emptyset \wedge \big(\forall p : (s.pc(p) = pc\_idle) \wedge (s.seen\_in(p) = \ false) \wedge$

$(s.seen\_out(p) = false)\big)\}$,

$in(IntAut) \ = \ \{findInv_p(k), insertInv_p(k), deleteInv_p(k) \ \mid \ k \ \in \ U, p \ \in$

$PROC\}$

$out(IntAut) = \{findResp_p(r), insertResp_p(r), deleteResp_p(r) \mid r \in boolean,$

$p \in PROC\}$

$int(IntAut) \ = \ \{doFindT_p(k), doFindF_p(k), doInsertT_p(k), doInsertF_p(k),$

$doDeleteT_p(k), doDeleteF_p(k) \mid k \in U, p \in PROC\}$

Figure 4.4: Initial states and *actions* of the intermediate automaton $IntAut$ for a $\mathcal{SET}$ data type.

return $false$ if $seen\_in(p)$ is $true$. Such an insert can be linearized at the time k was in $keys$. Similarly, a $delete_p(k)$ can decide to return $false$ if $seen\_out(p)$ is $true$. It can be linearized at the time k was not in $keys$. An $insert(k)$ of process p that returns $true$ not only adds the value into the abstract key set, but also sets the value of $seen\_in(q)$ to be true for any process q that is performing either $find(k)$ or $insert(k)$. Even if the key k is deleted later by some operation, by applying these changes, such a $find(k)$ or $insert(k)$ is allowed to return $true$ or $false$, respectively. Similarly, a $delete(k)$ of process p which returns $true$ removes the value from the

101 $bsr(i,a) \equiv (i.keys = a.keys)$

102 $\quad$ AND $\forall p : \Big[\ i.pc(p) = a.pc(p)$

103 $\qquad$ OR $\big(i.pc(p) = pcDoFind(k)$ AND

104 $\qquad\quad a.pc(p) = pcFindResp(false)$ AND $i.seen\_out(p) = true\big)$

105 $\qquad$ OR $\big(i.pc(p) = pcDoFind(k)$ AND

106 $\qquad\quad a.pc(p) = pcFindResp(true)$ AND $i.seen\_in(p) = true\big)$

107 $\qquad$ OR $\big(i.pc(p) = pcDoInsert(k)$ AND

108 $\qquad\quad a.pc(p) = pcInsertResp(false)$ AND $i.seen\_in(p) = true\big)$

109 $\qquad$ OR $\big(i.pc(p) = pcDoDelete(k)$ AND

110 $\qquad\quad a.pc(p) = pcDeleteResp(false)$ AND $i.seen\_out(p) = true\big)\Big]$

Figure 4.5: The backward simulation relation bsr between $IntAut$ and $AbsAut$.

abstract key set, and sets the value of $seen\_out(q)$ to be true for any process q that is performing either $find(k)$ or $delete(k)$, thereby allowing such a $find(k)$ or $delete(k)$ to return $false$, even if key k is inserted into the BST later by some operation.

After defining the intermediate automaton $IntAut$, we construct a backward simulation relation between them as follows. For $i \in state(IntAut)$ and $a \in state(AbsAut)$ we define bsr shown in Figure 4.5.

As we can see from the definition, bsr contains two parts. The first part (Line 101) requires that the $data$ (i.e., the keys set) of the related states of $IntAut$

and *AbsAut* should be identical, and the second part (Line 102-110) requires that the *Pcval* of each process p in *IntAut* stays "in step" with process p in *AbsAut*, with four exceptions. For example, Line 103 and 104 say that in *AbsAut*, p may already have executed *doFindF*, indicating that p's *find* operation will subsequently return *false*, whereas in *IntAut*, p is still processing the *find* operation and has not yet decided to return *false*. This is allowed only if *seen_out*(p) is *true*, which means either k is not in the key set at the invocation of the *find* operation, or is present at the invocation but is subsequently successfully deleted by some other process q before the *doFindF* is performed. The other three cases are similar.

When we construct the execution sequence of *AbsAut* in the backward simulation, for each action of *IntAut*, we choose the same action for the *AbsAut*, with the following exceptions. Intuitively, a $find_p(k)$ operation that returns *true* is linearized either at the time when the search begins (if key k is in the BST at the beginning of $find_p(k)$) or at the time immediately after some other operation successfully inserts k (if key k is not in the BST at the beginning of $find_p(k)$). At least one of those situations must be applicable, because *seen_in*(p) must be *true* before performing a $doFindT_p(k)$ in *IntAut*. Hence, key k is either in the BST at the beginning of a $find_p(k)$ or k is inserted by some other operations during the $find_p(k)$. Accordingly, when a $findInv(k, p)$ action is performed in *IntAut*, we choose a sequence of actions containing the same $findInv(k, p)$ action in *AbsAut*. This sequence in

$AbsAut$ may also contain a $doFindT(k, p)$ action immediately after the invocation, if $seen\_in(p)$ is $true$ in $IntAut$ and the $find_p(k)$ operation subsequently returns $true$. We know the future behaviour of an operation, because it is a backward simulation. Figure 4.6 shows an example.

In the other case, when the $find_p(k)$ operation subsequently returns $true$ in the future, but after the invocation of the $find_p(k)$ $seen\_in(p)$ is $false$, we linearize $doFindT(k, p)$ immediately after a $doInsertT(k, q)$ by some process q. Therefore, when a $doInsertT(k, p)$ action that successfully adds k into the key set in $IntAut$ occurs, we may choose a sequence of actions not only containing the same $doInsertT(k, p)$ action in $AbsAut$, but also followed by one $doFindT(k, q)$ action for each q that is executing a $find(k)$ operation that subsequently returns $true$ in the post state of $AbsAut$. Figure 4.7 shows an example of this case.

Figure 4.6 and 4.7 illustrate examples of how we construct states and actions in $AbsAut$ step by step starting from the end of the execution. In Figure 4.6, the $doFindT(3, p)$ action in $IntAut$ is linearized immediately after its invocation, because key 3 is in the BST at the invocation. However, in Figure 4.7, we cannot do the same thing, because the post state of $findInv(3, p)$ in $AbsAut$ indicates that $find(3)$ will not subsequently return $true$. Note that when $doInsertT(3, q)$ occurs in $IntAut$, we choose a sequence of actions containing the same $doInsertT(3, q)$ action in $AbsAut$, followed by one $doFindT(3, p)$ action for process p because it is

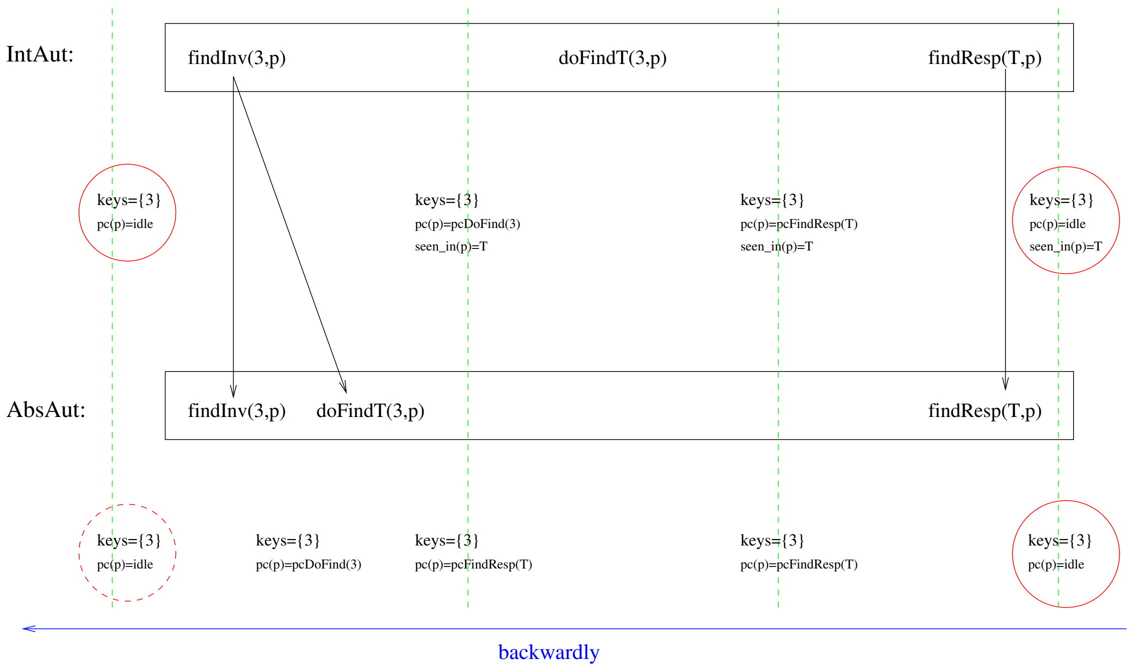


Figure 4.6: A simple example of how the backward simulation bsr works between $IntAut$ and $AbsAut$. Circles are known states and the dashed circle is the state constructed backwardly according to actions taken in $AbsAut$.

executing a $find(k)$ operation that subsequently returns $true$ according to the post state of $doInsertT(3, q)$ in $AbsAut$. It is also important to see that states paired by the green dotted lines satisfy bsr.

Similarly, a $find_p(k)$ operation that returns $false$ is linearized either at the time when the *search* begins (if key k is not in the BST at the beginning of $find_p(k)$) or at the time immediately after some other operation successfully deletes k (if key k is in the BST at the beginning of $find_p(k)$). At least one of those situations must be applicable, because $seen\_out(p)$ must be $true$ before performing a $doFindF_p(k)$

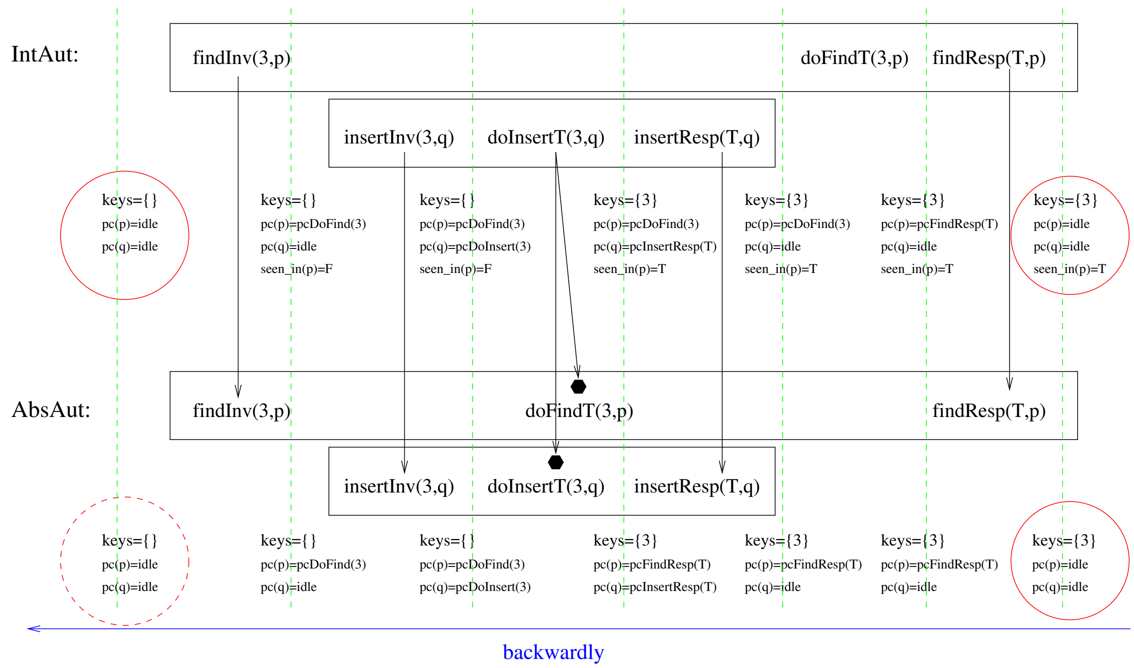


Figure 4.7: Another example of how the backward simulation *bsr* works between *IntAut* and *AbsAut*.

in *IntAut*. Hence, key k is either not in the BST at the beginning of the $find_p(k)$ or k is deleted by some other operation during the $find_p(k)$. Therefore, when a $findInv(k, p)$ action is performed in *IntAut*, we choose a sequence of actions containing the same $findInv(k, p)$ action in *AbsAut*. In addition, the sequence contains a $doFindF(k, p)$ action immediately after the invocation, if *seen_out*(p) is *true* in *IntAut* and the $find_p(k)$ subsequently returns *false*. Otherwise, if $find_p(k)$ operation subsequently returns *true*, but *seen_out*(p) is *false* when the find is invoked, we linearize $doFindT(k, p)$ immediately after a $doDeleteT(k, q)$ by some process q. Consequently, when a successful $doDeleteT(k, p)$ action in *IntAut* occurs, we may choose a sequence of actions not only containing the same $doDeleteT(k,p)$ action in $AbsAut$, but also followed by one $doFindF(k,q)$ action for each q that is executing a $find(k)$ operation that subsequently returns $false$ according to the post state of $AbsAut$.

Similarly, an $insert_p(k)$ (or a $delete_p(k)$) that returns $false$ may be linearized at the time immediately after its invocation or immediately after some other successful $doInsertT(k,q)$ (or $doDeleteT(k,q)$), depending on the value of $seen\_in(p)$ ($seen\_out(p)$).

To summarize: when a $doInsertT(k,p)$ action in $IntAut$ occurs, we may choose a sequence of actions not only containing the same $doInsertT(k,p)$ action in $AbsAut$, but also followed by one $doFindT(k,q)$ action for each q that is executing a $find(k)$ operation that subsequently returns $true$ and one $doInsertF(k,q)$ action for each q that is executing an $insert(k)$ that subsequently returns $false$ according to the post state of $AbsAut$. When a successful $doDeleteT(k,p)$ action in $IntAut$ occurs, we may choose a sequence of actions not only containing the same $doDeleteT(k,p)$ action in $AbsAut$, but also followed by one $doFindF(k,q)$ action for each q that is executing a $find(k)$ operation that subsequently returns $true$ according to the post state of $AbsAut$, and one $doDeleteF(k,q)$ action for each q that is executing a $delete(k)$ operation that subsequently returns $false$ according to the post state of $AbsAut$. Because we already linearized $doFindT(k,p)$, $doInsertF(k,p)$,

$doFindF(k,p)$ and $doDeleteF(k,p)$ actions of $AbsAut$, when any of these actions are performed in $IntAut$, they are ignored (i.e., we do not choose any action in $AbsAut$ for those four types of actions in $IntAut$). This is the action correspondence between $IntAut$ and $AbsAut$.

By using the bsr relation and our explicit construction of the action correspondence, we were able to show that a backward simulation exists between $IntAut$ and $AbsAut$. We have formalized the proof of this backward simulation using PVS.

4.4 The Forward Simulation

We also construct a forward simulation fsr from $ConcAut$ to $IntAut$. Firstly, we describe the action correspondence of the forward simulation. Most internal actions in $ConcAut$ correspond to the empty sequence (ϵ) of $IntAut$, except for some key actions shown in Table 4.8.

Intuitively, successful $ichild$ CASs ($insert14T(p)$ and $insert15T(p)$) starting from a state c in $ConcAut$ are mapped to $doInsertT(c.k(p),p)$ in $IntAut$, because both of these actions insert key $c.k(p)$ into the BST. Successful $dchild$ CASs ($delete17T(p)$ and $delete18T(p)$) starting from c in $ConcAut$ are mapped to $doDeleteT(c.k(p),p)$ in $IntAut$, since these actions delete key $c.k(p)$ from the BST. If an $insert4T(p)$ or $find2T(p)$ starting from c is performed in $ConcAut$, we shall prove that the given key $c.k(p)$ has been in the BST at some time since the beginning of the invocation, which is similar to the pre-condition for performing an $doInsertF(c.k(p), p)$ or $doFindT(c.k(p), p)$ in $IntAut$. If a $delete3T(p)$ or $find2F(p)$ is performed, we shall prove there was a time since the beginning of the invocation when the given key $c.k(p)$ was not in the BST. Hence, these two actions can be mapped to $doDeleteF(c.k(p), p)$ or $doFindF(c.k(p), p)$, respectively. Each external action of $ConcAut$ is mapped to its counterpart in $IntAut$.

Once again, fsr consists of a *data* relationship and a *Pcval* relationship. However, since it is not convenient for us to relate the concrete data structure of $ConcAut$ to the abstract set in $IntAut$ directly, we add an auxiliary variable *aux_keys* to the state of $ConcAut$ to represent all current keys in the BST. Therefore, the data relation part in fsr can simply require that *aux_keys* of $ConcAut$ is the same as *keys* of $IntAut$ if we establish as an invariant that *aux_keys* matches the set of all keys in the leaves of the BST. More specifically, *aux_keys* is updated as follows.

> *aux_keys*: Intuitively, this variable denotes all keys in the reachable leaves of the BST in $ConcAut$. Initially, $aux\_keys = \emptyset$. The new key k is added if a successful *ichild CAS* of $insert_p(k)$ operation is performed. The key k is removed if a successful *dchild CAS* of $delete_p(k)$ operation is performed.

Similarly, if there is a transition $c \xrightarrow{\alpha}_{ConcAut} c'$ and a state i in $IntAut$, such that $(c, i) \in fsr$, it is not convenient to reason about the value of $i.seen\_in$ or $i.seen\_out$ directly from the given state of $ConcAut$. We thus introduce *aux_seen_in* and

aux_seen_out into the state of $ConcAut$ and prove some invariants about them. The auxiliary variables aux_seen_in and aux_seen_out are updated as follows.

1. aux_seen_out: [PROC $\rightarrow$ *bool*]. The variable $aux\_seen\_out(p)$ is set to $true$ or $false$ according to whether $k \notin aux\_keys$ or not when a $findInv(k, p)$ or $deleteInv(k, p)$ action is performed. When a successful *dchild CAS* ($delete17T(p)$ or $delete18T(p)$) is performed at a state c and $c.k(p) = k$, then for each process q such that *aux_out_affected(q,c)* (see Figure 4.8) is $true$ and the given key of q is equal to k, $aux\_seen\_out(q)$ is set to $true$.

2. aux_seen_in: [PROC $\rightarrow$ *bool*]. The variable $aux\_seen\_in(p)$ is set to $true$ or $false$ according to whether $k \in aux\_keys$ or not when a $findInv(k, p)$ or $insertInv(k, p)$ action is performed. When a successful *ichild CAS* ($insert14T(p)$ or $insert15T(p)$) is performed at state c and $c.k(p) = k$, then for each process q such that $aux\_in\_affected(q, c)$ (see Figure 4.9) is $true$ and the given key of q is equal to k, $aus\_seen\_in(q)$ is set to $true$.

The function $aux\_in\_affected(p, c)$ is evaluated to be true, if process p is either in Line F1-F2 or Line I1-I15 of the simplified algorithm or is performing a $search(v)$ subroutine which is not invoked by a *delete* operation. Note that, p is considered to have completed an *insert* operation, if it is performing Line I17 or I18. Hence, $aux\_seen\_in(p)$ cannot be affected. Likewise, *aux_out_affected(p,c)* is evaluated to be true, if process p is either in Line F1-F2 or Line D1-D18 or Line D20 of the simplified

$$
\begin{aligned}
aux\_in\_affected(p,c) \;\equiv\;& c.pc(p) \in \{\, pcFind1, pcFind2,\\
& \quad pcInsert1, pcInsert2, pcInsert3, pcInsert4, pcInsert5,\\
& \quad pcInsert6, pcInsert7, pcInsert8, pcInsert9, pcInsert10,\\
& \quad pcInsert11, pcInsert12, pcInsert13, pcInsert14, pcInsert15 \,\}\\
& \text{OR } \Big(c.pc(p) \in \{\, pcSearch1, pcSearch2, pcSearch3, pcSearch4, pcSearch5,\\
& \quad pcSearch6, pcSearch7, pcSearch8, pcSearch9 \,\}\\
& \quad \text{AND } c.ret\_addr \neq pcDelete2 \Big)
\end{aligned}
$$

Figure 4.8: If an action starting from c, is an *ichild CAS*, $aux\_seen\_in(p)$ may be set to true for all p if $aux\_in\_affected(p,c)$ is true.

algorithm or is performing a $search(v)$ subroutine which is not invoked by an *insert* operation. Note that, p is considered to have completed a *delete* operation if it has performed a successful *dchild CAS*. *ConcAut* updates $aux\_seen\_in(p)$ at state c for key k, if $aux\_in\_affected(p,c)$ is true and the given key of stored in process p is the same to the key inserted by an *ichild CAS*. Symmetrically, we update aux_seen_out at state c to *true* for some process p and k, if $aux\_in\_affected(p,c)$ is *true* and $c.k(p) = k$. Hence, by requiring that $c.aux\_seen\_in$ and $c.aux\_seen\_out$ be the same as their counterparts in a state i of *IntAut* if $(c,i) \in fsr$ as part of the data relation part of fsr, we can prove the forward simulation holds between the two automata more conveniently.

$$
\begin{aligned}
aux\_in\_affected(p,c) \;\equiv\;& c.pc(p) \in \{\, pcFind1, pcFind2, \\
& pcDelete1, pcDelete2, pcDelete3, pcDelete4, pcDelete5, \\
& pcDelete6, pcDelete7, pcDelete8, pcDelete9, pcDelete10, \\
& pcDelete11, pcDelete12 pcDelete13, pcDelete14, pcDelete15, \\
& pcDelete16, pcDelete17, pcDelete18, pcDelete20 \,\} \\
\text{OR} \Big(\; & c.pc(p) \in \{\, pcSearch1, pcSearch2, pcSearch3, pcSearch4, pcSearch5, \\
& pcSearch6, pcSearch7, pcSearch8, pcSearch9 \,\} \\
& \text{AND } c.ret\_addr \neq pcInsert3 \Big)
\end{aligned}
$$

Figure 4.9: If an action starting from c, is a *dchild CAS*, $aux\_seen\_out(p)$ may be set to true for all p if *aux_out_affected(p,c)* is true.

Thus, for $c \in states(ConcAut)$ and $i \in states(IntAut)$, the *data* relationship between $ConcAut$ and $IntAut$ is stated as follows.

$$
\begin{aligned}
fsr\_data\_rel(c,i) \equiv (c.aux\_keys = i.keys) \wedge (c.aux\_seen\_in = i.seen\_in) \\
\wedge (c.aux\_seen\_out = i.seen\_out). \qquad (4.1)
\end{aligned}
$$

In addition to the data relation in fsr, there is also a program counter relation. For $c \in state(ConcAut)$ and $i \in state(IntAut)$, the program counter relation is defined in Figure 4.10. Relation $fsr\_pc\_rel\_find(c,i,p)$ describes that p is performing a $find$ operation in $ConcAut$ and p is performing a corresponding

$fsr\_pc\_rel(c, i) \equiv \forall$ p: $fsr\_pc\_rel\_find(c, i, p)$

OR $fsr\_pc\_rel\_insert(c, i, p)$

OR $fsr\_pc\_rel\_delete(c, i, p)$

OR (inSearch(c, p) AND $c.ret\_addr_p = pcFind2$ AND $i.pc_p = pcDoFind(c.k_p)$)

OR (inSearch(c, p) AND $c.ret\_addr_p = pcInsert3$ AND $i.pc_p = pcDoInsert(c.k_p)$)

OR (inSearch(c, p) AND $c.ret\_addr_p = pcDelete2$ AND $i.pc_p = pcDoDelete(c.k_p)$)

Figure 4.10: Program counter relation of fsr.

$find$ operation in $IntAut$. This relation is also shown in Figure 4.11. Relations $fsr\_pc\_rel\_insert(c, i, p)$ and $fsr\_pc\_rel\_delete(c, i, p)$ describing the program counter relations for an *insert* and *delete* operation are defined in a similar way. They are shown in Figure 4.13 and Figure 4.14, repectively. Details of the three relations *fsr_pc_rel_find(c,i,p)*, *fsr_pc_rel_insert(c,i,p)* and *fsr_pc_rel_delete(c,i,p)* are shown in Figure 4.12.

Intuitively, if $(c, i) \in fsr\_pc\_rel$, then for every process p, $fsr\_pc\_rel\_find(c, i, p)$ is satisfied, or $fsr\_pc\_rel\_insert(c, i, p)$ is satisfied, or $fsr\_pc\_rel\_delete(c, i, p)$ is satisfied. Otherwise, it is in the case that process p is performing a *search* subroutine as shown in Figure 4.15: if the *search* is invoked by a $find$ operation in $ConcAut$ then $i.pc_p = pcDoFind(c.k_p)$ is *true* in $IntAut$, or if the *search* is invoked by an *insert* operation in $ConcAut$ then $i.pc_p = pcDoInsert(c.k_p)$ is *true* in $IntAut$,

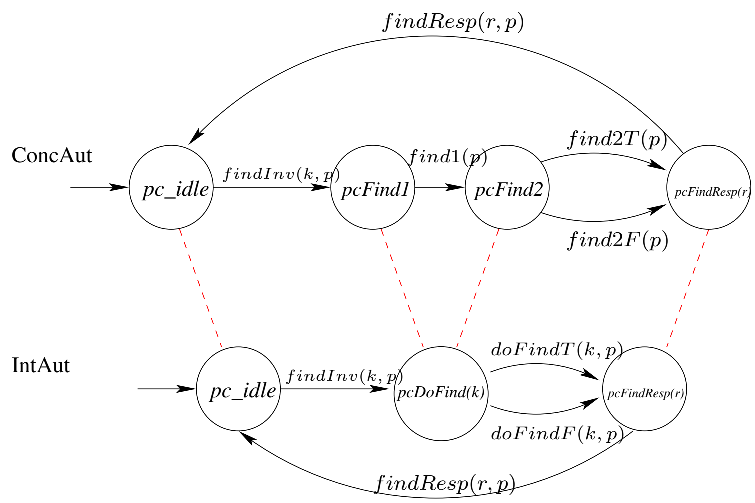


Figure 4.11: Program counter relation during the $find_p(k)$ operation in the forward simulation relation.

otherwise (the *search* is invoked by a *delete* operation), $i.pc_p = pcDoDelete(c.k_p)$ should be *true* in $IntAut$. Overall, a forward simulation relation fsr is defined as follows.

$$fsr(c, i) \ \equiv \ fsr\_data\_rel(c, i) \wedge fsr\_pc\_rel(c, i).$$

Action	**Precondition**	**Effect**
$findInv(k,p)$	s.pc(p) = $idle$	s.pc(p) $\leftarrow$ $pcFind1$ s.k(p) $\leftarrow$ k aux_seen_in(p) $\leftarrow$ ($k \in s.aux\_keys$) aux_seen_out(p) $\leftarrow$ ($k \notin s.aux\_keys$)
$find1(p)$	s.pc(p) = $pcFind1$	s.pc(p) $\leftarrow$ $pcSearch1$ s.ret_addr(p) $\leftarrow$ $pcFind2$
$find2T(p)$	s.pc(p) = $pcFind2$ AND s.keyf(s.ln(p)) = s.k(p)	s.pc(p) $\leftarrow$ $pcFindResp(true)$
$find2F(p)$	s.pc(p) = $pcFind2$ AND s.keyf(s.ln(p)) $\neq$ s.k(p)	s.pc(p) $\leftarrow$ $pcFindResp(false)$
$findResp(r,p)$	s.pc(p) = $pcFindResp(r)$	s.pc(p) $\leftarrow$ $idle$
$insertInv(k,p)$	s.pc(p) = $idle$	s.pc(p) $\leftarrow$ $pcInsert1$ aux_seen_in(p) $\leftarrow$ ($k \in s.aux\_keys$) s.k(p) $\leftarrow$ k
$insert1(p)$	s.pc(p) = $pcInsert1$	s.pc(p) $\leftarrow$ $pcInsert2$ s.nNode(p) $\leftarrow$ newNode($s, p, s.k(p)$)
$insert2(p)$	s.pc(p) = $pcInsert2$	s.pc(p) $\leftarrow$ $pcSearch1$ s.ret_addr(p) $\leftarrow$ $pcInsert3$
$insert14T(p)$	s.pc(p) = $pcInsert14$ AND s.lnf(s.op(p)) = s.leftf(s.pnf(s.op(p)))	s.pc(p) $\leftarrow$ $pcInsert16$ s.aux_keys.add($s.k(p)$) FOR EACH $q \in PROC$: IF (s.k(q) = s.k(p) AND aux_in_affected(q,s)) THEN s.aux_seen_in(q) $\leftarrow$ $true$
$insertResp(r,p)$	s.pc(p) = $pcInsertResp(r)$	s.pc(p) $\leftarrow$ $idle$
$deleteInv(k,p)$	s.pc(p) = $idle$	s.pc(p) $\leftarrow$ $pcDelete1$ aux_seen_out(p) $\leftarrow$ ($k \notin s.aux\_keys$) s.k(p) $\leftarrow$ k
$delete9(p)$	s.pc(p) = $pcDelete9$	s.pc(p) $\leftarrow$ $pcDelete10$ s.op1(p) $\leftarrow$ newMinfo($s, p, s.op1(p)$)
$delete17T(p)$	s.pc(p) = $pcDelete17$ AND s.pnf(s.op1(p)) = s.leftf(s.gpnf(s.op1(p)))	s.pc(p) $\leftarrow$ $pcDelete19$ s.aux_keys.remove($s.k(p)$) FOR EACH $q \in PROC$: IF (s.k(q) = s.k(p) AND aux_out_affected(q,s)) THEN s.aux_seen_out(q) $\leftarrow$ $true$
$deleteResp(r,p)$	s.pc(p) = $pcDeleteResp(r)$	s.pc(p) $\leftarrow$ $idle$

Table 4.6: Some transitions for the concrete automaton $ConcAut$, where s is a $state$ of $ConcAut$.

Action	**Precondition**	**Effect**
$findInv(k, p)$	$s.\mathrm{pc}(p) = idle$	$s.\mathrm{pc}(p) \leftarrow pcDoFind(k)$ $\mathrm{seen\_in}(p) \leftarrow (k \in \mathrm{s}.keys)$ $\mathrm{seen\_out}(p) \leftarrow (k \notin \mathrm{s}.keys)$
$doFindT(k, p)$	$s.\mathrm{pc}(p) = pcDoFind(k)$ AND $s.\mathrm{seen\_in}(p)$	$s.\mathrm{pc}(p) \leftarrow pcFindResp(true)$
$doFindF(k, p)$	$s.\mathrm{pc}(p) = pcDoFind(k)$ AND $s.\mathrm{seen\_out}(p)$	$s.\mathrm{pc}(p) \leftarrow pcFindResp(false)$
$findResp(r, p)$	$s.\mathrm{pc}(p) = pcFindResp(r)$	$s.\mathrm{pc}(p) \leftarrow idle$
$insertInv(k, p)$	$s.\mathrm{pc}(p) = idle$	$s.\mathrm{pc}(p) \leftarrow pcDoInsert(k)$ $\mathrm{seen\_in}(p) \leftarrow (k \in \mathrm{s}.keys)$
$doInsertT(k, p)$	$s.\mathrm{pc}(p) = pcDoInsert(k)$ AND $k \notin s.keys$	$s.\mathrm{pc}(p) \leftarrow pcInsertResp(true)$ $s.keys.\mathrm{add}(k)$ FOR EACH $q \in PROC$: IF $\big(s.\mathrm{pc}(q) = pcDoFind(k)$ OR $s.\mathrm{pc}(q) = pcDoInsert(k)\big)$ THEN $s.\mathrm{seen\_in}(q) \leftarrow true$
$doInsertF(k, p)$	$s.\mathrm{pc}(p) = pcDoInsert(k)$ AND $s.\mathrm{seen\_in}(p)$	$s.\mathrm{pc}(p) \leftarrow pcInsertResp(false)$
$insertResp(r, p)$	$s.\mathrm{pc}(p) = pcInsertResp(r)$	$s.\mathrm{pc}(p) \leftarrow idle$
$deleteInv(k, p)$	$s.\mathrm{pc}(p) = idle$	$s.\mathrm{pc}(p) \leftarrow pcDoDelete(k)$ $\mathrm{seen\_out}(p) \leftarrow (k \notin \mathrm{s}.keys)$
$doDeleteT(k, p)$	$s.\mathrm{pc}(p) = pcDoDelete(k)$ AND $k \in s.keys$	$s.\mathrm{pc}(p) \leftarrow pcDeleteResp(true)$ $s.keys.\mathrm{remove}(k)$ FOR EACH q: IF $\big(s.\mathrm{pc}(q) = pcDoFind(k)$ OR $s.\mathrm{pc}(q) = pcDoDelete(k)\big)$ THEN $s.\mathrm{seen\_out}(q) \leftarrow true$
$doDeleteF(k, p)$	$s.\mathrm{pc}(p) = pcDoDelete(k)$ AND $s.\mathrm{seen\_out}(p)$	$s.\mathrm{pc}(p) \leftarrow pcDeleteResp(false)$
$deleteResp(r, p)$	$s.\mathrm{pc}(p) = pcDeleteResp(r)$	$s.\mathrm{pc}(p) \leftarrow idle$

Table 4.7: Transitions for the intermediate automaton $IntAut$, where s is the variable of TYPE *state* of $IntAut$.

Actions in $ConcAut$ **starting from** c		**Actions in** $IntAut$
$find2T(p)$		$doFindT(c.k(p), p)$
$find2F(p)$		$doFindF(c.k(p), p)$
$insert4T(p)$		$doInsertF(c.k(p), p)$
$insert14T(p)$ $insert15T(p)$		$doInsertT(c.k(p), p)$
$delete3T(p)$		$doDeleteF(c.k(p), p)$
$delete17T(p)$ $delete18T(p)$		$doDeleteT(c.k(p), p)$
other actions		ϵ

Table 4.8: Internal action correspondence between $ConcAut$ and $IntAut$.

$$
\begin{aligned}
fsr\_pc\_rel\_find(c,i,p) \equiv\ & \Big(\ i.pc(p) = pcDoFind(c.k_p) \\
& \quad \text{AND } c.pc(p) \in \{pcFind1, pcFind2\}\Big) \\
& \text{OR } \Big(\ i.pc(p) = pcFindResp(r) \\
& \quad \text{AND } c.pc(p) = pcFindResp(r)\Big)
\end{aligned}
$$

$$
\begin{aligned}
fsr\_pc\_rel\_insert(c,i,p) \equiv\ & \Big(\ i.pc(p) = pcDoInsert(c.k_p) \\
& \quad \text{AND } c.pc(p) \in \{pcInsert1, \cdots, pcInsert15\}\Big) \\
& \text{OR } \Big(\ i.pc(p) = pcInsertResp(true) \\
& \quad \text{AND } c.pc(p) \in \{pcInsert16, pcInsert17\}\Big) \\
& \text{OR } \Big(\ i.pc(p) = pcInsertResp(r) \\
& \quad \text{AND } c.pc(p) = pcInsertResp(r)\Big)
\end{aligned}
$$

$$
\begin{aligned}
fsr\_pc\_rel\_delete(c,i,p) \equiv\ & \Big(\ i.pc(p) = pcDoDelete(c.k_p) \\
& \quad \text{AND } c.pc(p) \in \{pcDelete1, \cdots, pcDelete18, pcDelete20\}\Big) \\
& \text{OR } \Big(\ i.pc(p) = pcDeleteResp(true) \\
& \quad \text{AND } c.pc(p) = pcDelete19\Big) \\
& \text{OR } \Big(\ i.pc(p) = pcDeleteResp(r) \\
& \quad \text{AND } c.pc(p) = pcDeleteResp(r)\Big)
\end{aligned}
$$

Figure 4.12: Program counter relation of fsr when p is performing a $find$, $insert$ or $delete$ operation.

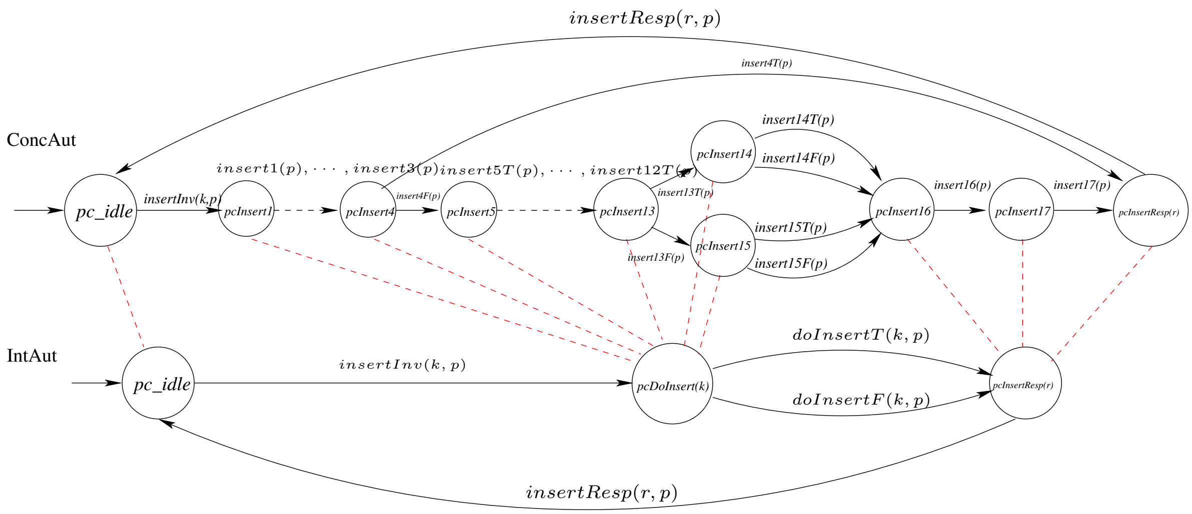


Figure 4.13: Program counter relation during the $insert_p(k)$ operation in the forward simulation relation.

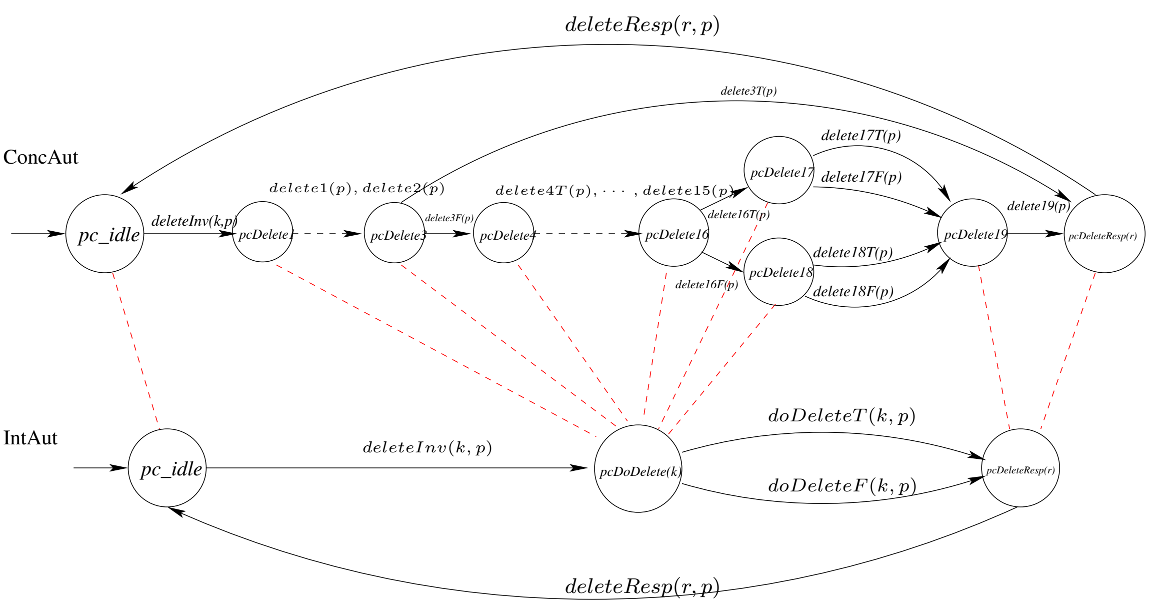


Figure 4.14: Program counter relation during the $delete_p(k)$ operation in the forward simulation relation.

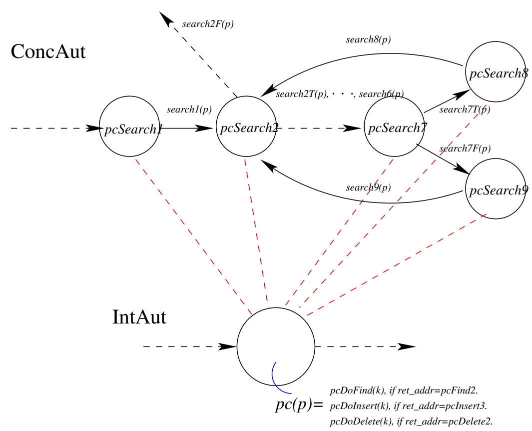


Figure 4.15: Program counter relation during the $search_p(k)$ subroutine in the forward simulation relation.

5 Invariants and Proofs

5.1 An Overview of the Proof

In Chapter 4, we described the way to model the specification and the simplified algorithm using a canonical automaton $AbsAut$ and a concrete automaton $ConcAut$, respectively. Because it is complicated to construct a forward simulation directly from $ConcAut$ to $AbsAut$, we introduced the intermediate automaton $IntAut$ and proved that $ConcAut$ implements $IntAut$ and $IntAut$ implements $AbsAut$ through forward and backward simulations, respectively. This hybrid forward and backward simulation implies that $ConcAut$ implements $AbsAut$, and hence our simplified algorithm satisfies its specification.

We have already defined the forward simulation fsr and the backward simulation bsr in Chapter 4. For the forward simulation (Definition 2.19), we have to prove three main properties.

1. For every $c \in start(ConcAut)$, there exists an $i \in start(IntAut)$, such that $(c, i) \in fsr$.

2. If $c \xrightarrow{\alpha}_{ConcAut} c'$ and $(c, i) \in fsr$, then there exists $\hat{\alpha}$ and i' such that $i \overset{\hat{\alpha}}{\Longrightarrow}_{IntAut} i'$ and $(c', i') \in fsr$, and

3. the external action in $\hat{\alpha}$ is the same as the external action in α.

The first condition requires that every initial state (in Section 4.2) in *ConcAut* has a matching initial state in *IntAut*. Because *IntAut* has a unique initial state (defined in Section 4.3), it was trivial to check that every initial state in *ConcAut* is related with the initial state in *IntAut* by *fsr* (defined in Section 4.4). The second property was proved by case analysis of all actions in *ConcAut*, using the action correspondence shown in Table 4.8. In most cases proving the second property was not complicated. The exceptions were the actions in *ConcAut* that map to non-nil actions of *IntAut*. These are, by far, the bulkiest part of the proof because they required proving many auxiliary lemmas about how the concrete automaton behaves. The last condition was straightforward to verify according to the action correspondence defined in Table 4.8.

Likewise, using Definition 2.22, which defines a backward simulation between *IntAut* and *AbsAut*, we proved that the following four properties of the backward simulation hold.

1. For all $i \in states(IntAut)$, there exists $a \in AbsAut$ such that $(i, a) \in bsr$.

2. If $i \in start(IntAut)$ and there exists $a \in states(AbsAut)$ such that $(i, a) \in bsr$, then $a \in start(AbsAut)$.

3. If $i' \xrightarrow{\alpha}_{IntAut} i$ and $a \in states(AbsAut)$ such that $(i, a) \in bsr$, then there exist a' and $\hat{\alpha}$ such that $(i', a') \in bsr$ and $a' \overset{\hat{\alpha}}{\Longrightarrow}_{AbsAut} a$, and

4. the external action in $\hat{\alpha}$ is the same as the external action in α.

The first property was proved by explicitly constructing a state of *AbsAut* from the state of *IntAut*. We proved the second property for the unique initial states for both *IntAut* and *AbsAut*. Once again, the third property was proved by enumerating all actions in *IntAut* using the action correspondence defined in the last part of Section 4.3. The difficult cases are $doInsertT(k, p)$ and $doDeleteT(k, p)$, whose corresponding actions in *AbsAut* may consist of several internal actions of other processes. We shall discuss that part of backward simulation proof in Section 5.5. After showing the third property, the last one is easy to prove using the action correspondence.

5.2 Proofs in the Forward Simulation

To complete the proof of the forward simulation defined in Chapter 4 between *ConcAut* and *IntAut*, we must show the key actions in Table 4.8 satisfy the following property.

> If $c \xrightarrow{\alpha}_{ConcAut} c'$ and $(c, i) \in fsr$, then there exist $\hat{\alpha}$ and i' such that $i \overset{\hat{\alpha}}{\Longrightarrow}_{IntAut} i'$ and $(c', i') \in fsr$

As we described in the overview section, given states $c, c' \in states(ConcAut)$ and $i' \in states(IntAut)$ and an action α such that $c \xrightarrow{\alpha}_{ConcAut} c'$ and $(c, i) \in fsr$, we can explicitly construct $\hat{\alpha}$ using the action correspondence between $ConcAut$ and $IntAut$ defined in Table 4.8. Hence, it remains to construct an $i' \in states(IntAut)$ which satisfies $i \xrightarrow{\alpha}_{IntAut} i'$ and $(c', i') \in fsr$.

Firstly, we prove that $i \xrightarrow{\alpha}_{IntAut} i'$ after constructing i' using $\hat{\alpha}$. For each action in $ConcAut$, we need to prove pre-state i enables $\hat{\alpha}$. A state i enables an action if the value of the program counter and the data values of i satisfy the precondition of the action defined in Table 4.7. Hence, we mainly focus on the cases which map to non-trivial actions in Table 4.8. If the action is a $find2T(p)$, to show there exists a $doFindT(c.k(p), p)$ action starting from i in $IntAut$, we need to argue $i.seen\_in(p)$ is $true$. This can be proved by showing $c.aux\_seen\_in(p)$ is $true$ since $i.seen\_in(p) = c.aux\_seen\_in(p)$ from $(c, i) \in fsr$. By the program counter relation part of fsr, because $c.pc(p) = pcFind2$, we have $i.pc(p) = pcDoFind(k)$. Therefore, preconditions of a $doFindT(k, p)$ action are satisfied in i. Symmetrically, we can prove that i satisfies the preconditions of a $doFindF(c.k(p), p)$ action in $IntAut$ when a $find2F(p)$ is performed in $ConcAut$. Hence, Lemma 5.1 is needed when a $find2T(p)$ or $find2F(p)$ action occurs in $ConcAut$.

Lemma 5.1. *Let c be any reachable state of ConcAut. If $c \xrightarrow{find2T(p)} c'$, then the value of $aux\_seen\_in_p$ is true at the state c. If $c \xrightarrow{find2F(p)} c'$ then the value of $aux\_seen\_out_p$*

is true at the state c.

A similar argument is applied to $insert4T(p)$ and $delete3T(p)$, as well. When a successful *ichild CAS* is performed, by an $insert14T(p)$ or $insert15T(p)$ action, let $k = c.k(p)$. We have to argue that $k \notin i.keys$ and $i.pc(p) = pcDoInsert(k)$ before the action $doInsertT(k,p)$ is taken in $IntAut$ according to Table 4.7. Since $i.keys = c.aux\_keys$ follows from $(c,i) \in fsr$, we just have to prove $k \notin c.aux\_keys$. Once again, because $c.pc(p) = pcInsert14$ or $c.pc(p) = pcInsert15$ before $insert14T(p)$ or $insert15T(p)$, respectively, $i.pc(p) = pcDoInsert(k)$ according to the program counter relation of $(c,i) \in fsr$. Similarly, when a successful *dchild CAS* is performed by a $delete17T(p)$ or $delete18T(p)$ action, let $k = c.k(p)$. We have to argue that $k \in c.aux\_keys$ and $i.pc(p) = pcDoDelete(k)$. Hence, we need to prove Lemma 5.2 and 5.3 for the ichild and dchild steps in $ConcAut$, respectively.

Lemma 5.2. *Let c be any reachable state of $ConcAut$. If $c \overset{insert4T(p)}{\longrightarrow} c'$ then the value of $aux\_seen\_in_p$ is true at the state c. If $c \overset{insert14T(p)}{\longrightarrow} c'$ or $c \overset{insert15T(p)}{\longrightarrow} c'$, then $c.k(p)$ is not in aux_keys at the state c.*

Lemma 5.3. *Let c be any reachable state of $ConcAut$. If $c \overset{delete3T(p)}{\longrightarrow} c'$, then the value of $aux\_seen\_out_p$ is true at the state c. If $c \overset{delete17T(p)}{\longrightarrow} c'$ or $c \overset{delete18T(p)}{\longrightarrow} c'$, then $c.k(p)$ is in aux_keys at the state c.*

Secondly, we need to show that after taking $\hat{\alpha}$ from i, the resulting state i' satisfies $(c', i') \in fsr$. Thus, for each action in $ConcAut$, we need to prove the data relation is satisfied between c' and i', as well as the program counter relation part of fsr. We again focus on the cases in Table 4.8, which are the most complicated ones. With respect to the data relation between $c'.aux\_keys$ and $i'.keys$, we know that aux_keys changes only if a successful child CAS is performed. The key k is added to aux_keys when a successful ichild CAS by process p that inserts key k occurs. Hence, in $IntAut$, a $doInsertT(k, p)$ action that adds k to $i.keys$ is performed. Because $c.aux\_keys = i.keys$, we have $c'.aux\_keys = i'.keys$. Symmetrically, $c'.aux\_keys = i'.keys$ holds if a successful dchild CAS that deletes k occurs.

To prove the set stored in the BST is the same as the set of keys in $IntAut$, we have to ensure that aux_keys is equal to the set of keys in the BST's reachable leaves in $ConcAut$. This is Invariant 1, which encapsulates the connection between the key set and its representation in shared memory as a BST. Hence, the complex structure of the BST in the concrete automaton is hidden by this auxiliary key set variable.

Invariant 1. *The set aux_keys in ConcAut always contains the same keys as the current reachable leaves in the tree starting from the Root node.*

With respect to the data relation between $c'.aux\_seen\_in$ and $i'.seen\_in$, and between $c'.aux\_seen\_out$ and $i'.seen\_out$, these parts of the states are initialized at each invocation of each operation and modified only during a successful child CAS. If α is an invocation by p, because $c.aux\_keys = i.keys$, $c'.aux\_seen\_in(p)$ is initialized to the same value as $i'.seen\_in(p)$, as are $c'.aux\_seen\_out(p)$ and $i'.seen\_out(p)$. When a successful *ichild CAS* by process p for key k is performed, $\hat{\alpha} = doInsertT(k, p)$, and $aux\_seen\_in(q)$ of every process q that is performing a $find(k)$ or an $insert(k)$ operation but has not yet decided to return *true* will be set to *true* in the post state c'. Hence, we also need to show that $i'.seen\_in(q)$ is *true* in order to prove $c'.aux\_seen\_in = i'.seen\_in$, which is required for showing $(c', i') \in fsr$. Since that process q is performing a $find(k)$ or an $insert(k)$ in *ConcAut*, it follows that q's program counter value is $pcDoFind(k, q)$ or $pcDoInsert(k, q)$ at state i in $IntAut$ due to the program counter relation of $(c, i) \in fsr$. Thus, according to the way that $IntAut$ updates variables in Table 4.7, $i'.seen\_in(q)$ is set to *true* by $\hat{\alpha}$ as well. In a symmetric way, when a successful *dchild CAS* occurs, we can show aux_seen_out in *ConcAut* is also related to $seen\_out$ in $IntAut$.

To prove the program counter relation holds between c' and i', we expand the effects of $\hat{\alpha}$ and show that after $\hat{\alpha}$, the program counter values of c' and i' are still related. There is one type of special case in proving the program counter relation between c' and i'. For example, when a failed dchild *CAS* ($delete17F(p)$) is performed, we know that $\hat{\alpha} = \epsilon$, and $(c, i) \in fsr$. However, we cannot relate c' to i', because $c.pc(p) = pcDelete19$ which is about to return, however, $i'.pc(p) = i.pc(p) = pcDoDelete(k)$. Therefore, we have to show $delete17F(p)$ cannot occur. This is because no helping mechanism is implemented, so it is impossible for a delete operation that successfully marks a parent node to fail on the dchild *CAS* or for an insert operation that successfully flags a parent node to fail on the ichild *CAS*. This is formalized in the following Lemma 5.4.

Lemma 5.4. *For any execution of $ConcAut$, if a process p successfully performs a mark* CAS *($delete10T(p)$), it cannot perform an unsuccessful dchild* CAS *($delete17F(p)$ or $delete18F(p)$ in the same iteration of the loop). Similarly, if p successfully performs an iflag* CAS *($insert11T(p)$), it cannot perform an unsuccessful ichild* CAS *($insert14F(p)$ or $insert15F(p)$ in the same iteration of the loop).*

Combined with the auxiliary variables $aux\_seen\_in(p)$ and $aux\_seen\_out(p)$, Lemma 5.1, 5.2 and 5.3 correspond to the Lemma 5.5 and 5.6 and 5.7, reproduced below, in the original English proof in the tech report [3].

Lemma 5.5. *If a $Find(k)$ operation returns true, then the BST contains a leaf with key k at some point between the beginning and end of the operation. If it returns false, there exist a time between the beginning and end of the operation such that the BST does not contain a leaf with key k.*

Lemma 5.6. *An $Insert(k)$ operation returns true if and only if the BST does not contain a leaf with key k just before it performs the ichild CAS. If the operation returns false, there exist a time between the beginning and end of the operation such that the BST contains a leaf with key k.*

Lemma 5.7. *A Delete(k) operation returns true if and only if the BST does contain a leaf with key k just before it performs the dchild CAS. If the operation returns false, there exist a time between the beginning and end of the operation such that the BST does not contain a leaf with key k.*

These lemmas, which require another 25 technical lemmas, were all proved in English in the original paper [3]. Thus, we formalized their proofs in PVS to complete the forward simulation. For example, we needed to prove one of the most important lemmas which claims that in any reachable state, the data structure maintained by the implementation is a BST, shown as Lemma 5.8.

Lemma 5.8. *In every reachable state, the tree of child pointers is a BST [3].*

This is also one of the key lemmas proved in the original paper ([3], Lemma 22).

We encountered some difficulties in formalizing lemmas written in [3] and formally proving them using PVS. An important difference between proofs written in natural language and machine checkable proofs is that a small step in the natural language proof in a human's mind is often not a straightforward automatic step in PVS. PVS provides some proof commands, such as *grind*, to automatically reason towards a goal. However those procedures, which try repeated skolemization, instantiation, and if-lifting, are not intelligent enough to prove complex goals, especially when some complicated data structures are involved. In proving the lemmas and invariants of the BST algorithms in PVS, one must be very careful of using such commands: if one uses the *grind* command carelessly, PVS automatically expands the definition of *ConcAut* into many cases and complicated expressions, which are very hard to work with. To avoid this, we have to explicitly state those "small" steps for humans as lemmas such that we can apply them when proving a higher level statement. Therefore, our PVS scriptto state the invariants of *ConcAut* contains many lemmas which may seem fairly trivial.

For instance, Lemma 4 in [3] states that for each internal node v, no *CAS* ever changes $v.info$ to a value that was previously stored there. In the proof, the authors state "Each successful flag *CAS* on $v.info$ subfield sets the filed to point to a newly created Info object, so that this object could never have appeared in $v.info$ before." This is easy to verify for a human: a successful iflag *CAS* (Line I11) is always preceded by a creation of a new Info object (Line I10). We may use another implicit fact that no other processes can write or modify the object between Line I10 and I11 because the only pointer to it is in a local variable of the process that created it. Therefore, it has never been visible to other processes before the successful iflag *CAS*. However, to verify that sentence using PVS, we have to split it into three small lemmas as follows.

First, we show that an Info object newly created on Line I10 has never appeared in $v.info$ before, as claimed by Lemma 5.9. The definition of executions in PVS are discussed in Section 5.3.

Lemma 5.9. *For any execution, if an Info object f is newly created by I10 at step i, any Info object accessed by any process before step i is not the same as f.*

This lemma can be proved by a contradiction. Assume that before step i, a process accessed an Info object that is the same as f. By applying Lemma 5.10 and Lemma 5.11, which claim that in any previous state, if x was an allocated Info object, it is still in the set *allocatedInfo*, we know that f is in *allocatedInfo*. By applying the axiom that describes the creation a new Info object, the object that I10 allocated must not be in *allocatedInfo*. Thus, it completes the proof.

Lemma 5.10. *For any execution, if a process accesses an Info object, this object has been allocated.*

Lemma 5.11. *For any execution, if f was in* allocatedInfo*, it is still in* allocatedInfo.

Second, Lemma 5.12 says that if process p creates an Info object f at I10, before p successfully performs its iflag *CAS* (I11) that writes a pointer to f into a node v, no other process can access f. This lemma can be proved by induction on the length of the execution.

Lemma 5.12. *For any execution and any node v, if an Info object f is newly created by I10 by process p at step j, and p points $v.info$ to f by a successful iflag* CAS *at step i, then no other process can access f between steps j and i.*

Third, we use Lemma 5.13 to state that for any successful iflag CAS performed by process p, there exists a previous execution of Line I10 by p such that no step is taken by p in between. Lemma 5.13 can be observed from the pseudocode. Hence, no process other than p can access f before the successful iflag CAS. It follows that no $v.info$ can be set by a CAS operation to point to f by any process other than p. Combined with Lemma 5.9, we can prove the single sentence written by the authors.

Lemma 5.13. *For any execution and any node v, if process p successfully changes $v.info$ by an iflag* CAS *to f at step i, there exists a step j such that $j < i$ and f is created by I10 performed by p and no step belongs to p in between j and i.*

Another kind of difficulty that appears when proving invariants of $ConcAut$ using PVS is what we call code structure problems. One may get a flavour of this problem by considering Lemma 5.13. Since we model the pseudocode as an I/O automaton, each line of the code becomes an independent action. In the pseudocode, one can easily observe how the code is executed line by line. But when we model it as an I/O automaton, proving something that relies on properties of the code structure, requires reasoning about many independent actions, which are tied together via their effects on the program counter. More specifically, if we know that process p is executing Line I13, we can easily conclude that the IF condition executed by p at Line I5 returns true, by looking at the pseudocode. However, in PVS, as the steps are modelled as independent actions, to prove the same statement, we have to state it as a lemma and infer step by step from Line I13 back to Line I5 and then conclude that if p executes Line I13, there exists an earlier step when p executed I5 and returned true. More generally, whenever one of this kind of situation occurs, we have to come up with an individual lemma to state it. This is quite inefficient and thus we want to build the kind of general tools for proving this kind of facts about an automaton that models code.

5.3 Some Definitions in PVS

Suppose $c \xrightarrow{\alpha} c'$, where c and c' are states and α is an action in *ConcAut*. As we can see, besides describing properties of c and c', we also want to reason about the previous actions before α or some possible actions after α in an execution. Many proofs of lemmas in the original paper use that kind of reasoning ([3], Lemma 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, etc.). Our first attempt to formalize a reachable state by an execution sequence starting from an initial state, adapted from [16], was inductive:

```
reachable?(c): INDUCTIVE bool =
               initial_config(c) ∨ (∃ s,α: reachable?(s) ∧ transition(s,α,c))
```

This says that a state c is reachable if it is an initial state or there is another state s which is reachable and there is a transition from s to c by performing an action α. This definition, although clear and simple, makes it awkward to reason about the actions before a given action or the actions that occur after it. Inspired by [28], we use a more natural way to define states and executions. We define a finite execution to be a sequence of $n+1$ states and n actions alternating with each other.

We define a *FiniteStepSeq* in PVS to be a type that consists of two finite sequences (finseq), one containing the actions and the other containing the states, where the length of the state-sequence is one larger than the length of the action-sequence. In PVS, the pair "[#" and "#]" represents a definition of a record type with several attributes. Its attributes, which consist of a name followed by a type, are separated by ",". The finseq type, provided by PVS, is a function that maps each natural number that is smaller than the length to an element of a generic type T.

```
finseq: TYPE = [# length: nat, seq: [ below[length] -> T ] #]
FiniteStepSeq: TYPE = [# actions: finseq[action],
                        states:{ss: finseq[state] | ss.length = actions.length+1} #]
```

If *stepseq* is a variable of type FiniteStepSeq, we define the function *steps*, which takes a FiniteStepSeq *stepseq* as its argument and returns a finite sequence of ⟨state, action, state⟩ tuples, *i.e.*, transitions.

```
stepseq: VAR FiniteStepSeq

steps(stepseq): finseq[ [state,action,state] ] =

       (# length := stepseq.actions.length,

          seq := λ.(n:below[stepseq.actions.length]):

                     (stepseq.states(n), stepseq.actions(n), stepseq.states(n+1))

        #)
```

A *stepseq* is a finiteExecFrag of an I/O automaton if and only if every tuple in *steps*(*stepseq*) is a legal transition of the I/O automaton. Furthermore, a finiteExecFrag *stepseq* is a finiteExecution if and only if the first state of *stepseq* is an initial state.

```
finiteExecFrag(stepseq): bool =

     ∀ (n: below[stepseq.actions.length]): transition(steps(stepseq)(n))

finiteExecution(stepseq): bool =

      finiteExecFrag(stepseq) ∧ initial_config(stepseq.state(0))
```

Hence, if $c \xrightarrow{\alpha} c'$ is the ith transition in a finiteExecution *stepseq*, we can easily reason about properties of any actions or states in the execution before ith step or after ith by referring to their indices. For example, Lemma 5.15 ([3], Lemma 9) can now be formalized using our definitions about finiteExecutions as shown in Figure 5.1, given the definition of "belong to" as follows.

Definition 5.14. A successful flag CAS belongs to an Info object f, if the flag CAS stores a pointer to an Info object f. A mark CAS belongs to an Info object f, if the $dinfof$ field of the Info object used by the CAS points to f.

Lemma 5.15. *Each mark* CAS *that belongs to an Info object f is preceded by a successful dflag* CAS *that belongs to f.*

Let $c \xrightarrow{\alpha} c'$. A frequently used lemma that requires proof in PVS is that every local variable of process p remains unchanged between c and c' if α belongs to a process $q \neq p$. When we prove an invariant, we always have to prove it is preserved by all possible actions α. There are 79 possible actions in $ConcAut$. However, except for a few important actions, most of the actions can be proved to preserve an invariant by using the same proof steps. Therefore, we usually construct PVS proof strategies, which consist of a batch of PVS proof commands, to automate the proofs when we need to enumerate all the actions.

5.4 Errors Found

Although proving the correctness of invariants and lemmas using PVS took a long time, we did detect some errors in the original proof. An author of [3] detected that the proof of Lemma 5.16 ([3], Lemma 2(10)) has a small error.

Lemma 5.16. *The top part of the tree is always as shown in Figure 5.2. More precisely:*

```
lemma_dflag_before_mark: LEMMA ∀ stepseq: finiteExecution(stepseq) ⇒

     ( ∀ i≤stepseq.actions.length:  LET alpha=stepseq.actions(i),

                                        p=process(stepseq.actions(i)),

                                        c=stepseq.states(i),

                                        f=c.dinfof(c.op2(p)) IN

        (markCAS(alpha) ⇒

              ∃ j:nat: j≤i ∧ success_dflagCAS_belong_f(stepseq,j,f)

         )

     )


markCAS(alpha): bool =  (alpha = delete10T OR alpha = delete10F)


success_dflagCAS_belong_f(stepseq,i,f): bool = LET s=stepseq.states(i),

                                                beta=stepseq.actions(i) IN

     (beta = delete7T AND s.op1(process(beta)) = f)
```

Figure 5.1: Using definition of finiteExecution to formalize a lemma.

(a) *Root.left.key* $= \infty_1$, *and*

(b) *if Root.left is an internal node, then Root.left.right is a leaf with key* ∞_1.

Their proof was done by induction on states in an execution. It is trivial that the lemma holds for the base case, where the state is the initial state. However, for the induction step, let $c \xrightarrow{\alpha} c'$ and assume the claim holds in c. For the case where α is a dchild CAS that changes the node $root.left$ using some Info object f, they argued that after the dchild CAS, $root.left$ is a leaf with key ∞_1, because $f.pn.right$ is a leaf with key ∞_1. Because the dchild CAS is successful, $root.left$ points to $f.pn$ in state c. They also have a lemma that proves that for any DInfo object f, $f.pn$ is an internal node. They claimed that, it follows from the induction hypothesis that $f.pn.right$ is a leaf with key ∞_1.

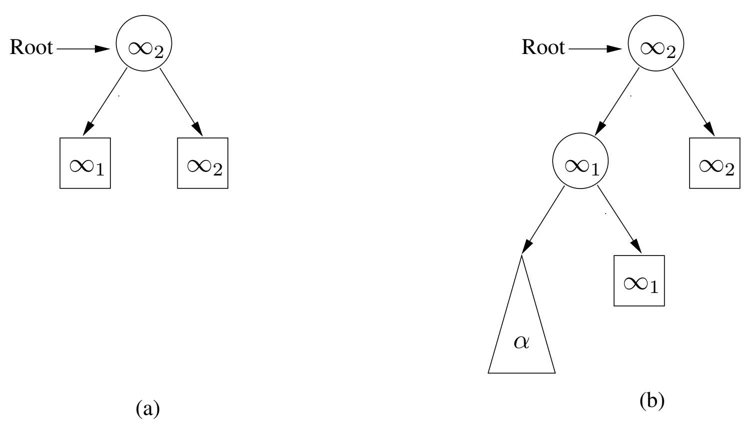


Figure 5.2: Trees showing leaves when the set is (a) empty and (b) non-empty. [3]

This is incorrect. We agree that $root.left$ is an internal node at c, and thus we can apply the hypothesis to show $root.left.right$ is a leaf with key ∞_1 in state

c. However, the dchild *CAS* writes the value stored in *other* to $root.left$, and the value of *other* was assigned at a step β before α. In the state immediately before β, we do not know if $f.pn.right$ at this point is a leaf with ∞_1, since we do not know that $f.pn$ is the left child of the *root* at that time. One way to fix the proof of this lemma is to use lemmas proved subsequently, which say that if a node is flagged or marked then no other process can modify its child pointers, and before a successful dchild *CAS*, the node is marked. However, in order to do so, all those lemmas have to be composed into a big induction lemma, which is a bit complicated. Instead, we fixed this lemma by making it a bit weaker as stated in Lemma 5.17.

Lemma 5.17. *The node $root.left$ is always a node with key ∞_1.*

This weaker lemma turns out to be sufficient to be used in the later proofs, since we can still conclude that $root.left.right$ has a key greater than or equal to ∞_1 by combining a few lemmas.

In the process of formalizing the proof, I discovered one flaw in the original proof in Lemma 5.18 ([3], Lemma 14(7)). Note that Lemma 14 in the original proof is a big induction lemma which has many parts and we mainly discuss the seventh part of it.

Lemma 5.18. *A child (either an ichild or dchild) CAS writes a value into a node v's child pointer that has never been stored there before.*

We can focus on the case of a dchild *CAS* $dcas_k$ that changes the left child of

x from a node z to y. In the original proof, to derive a contradiction, the authors assume that y was the left child of x at some earlier time. Figure 5.3 illustrates the execution for the proof. In the state just before $dcas_k$, z's child is y. Because we

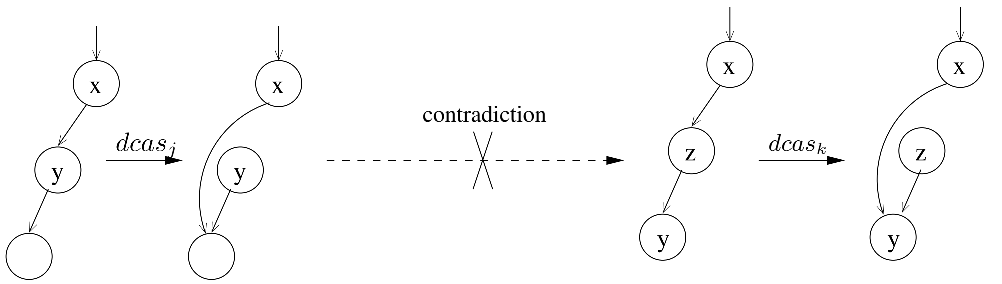


Figure 5.3: The way to show contradiction in proving Lemma 5.18.

know $y \neq z$, there must be an earlier child *CAS* $ccas_j$ that caused y to stop being the left child of x. They proved that just after $ccas_j$, y is not a descendant of x. The case where $ccas_j$ is an ichild *CAS* is fine. So, we only consider the case where $ccas_j$ is a *dchild CAS* as shown in Figure 5.3. According to another part of the induction hypothesis "before a *dchild CAS*, the child pointers of the parent node *f.pn* do not change between the last read in *search* belong to f and the *dchild CAS*", so $ccas_j$ replaces a pointer to y by a pointer to y's child, and y is no longer a descendant of x (since y cannot be a descendant of its own child by "the binary tree property" (another part of the induction hypothesis)). This is incorrect, because "the binary tree property" can only be applied here when y is reachable. However, there is no proof to show node x or y is reachable before the *dchild CAS*.

We fix this lemma by adding more auxiliary claims into the original Lemma 14 to compose a bigger induction lemma. The added claims are stated in Lemma 5.19. Some of these claims were proved in [3], but were not wrapped up in the induction proof used to prove Lemma 14 in [3]

Lemma 5.19. *1. After a successful dchild* CAS *by a process p, the node that was marked by p before the dchild* CAS *and was reachable right before the* CAS*, becomes unreachable and will never become reachable again.*

2. If a successful child (either an ichild or dchild) CAS occurs on a node v, v is reachable in the state right before the CAS*.*

3. During a search subroutine of process p, each visited node was reachable at a time before it is visited by p.

4. The node which is unreachable and becomes reachable by an ichild CAS was never reachable before.

5. If a node is reachable after any action other than a successful ichild CAS *then it was reachable before the action as well.*

Intuitively, Lemma 5.19(1) can be proved by two cases. In case 1, if a node is added by a successful *ichild CAS* we prove this node is not the marked node by contradiction by applying Lemma 5.19(4). In the other case, when a node becomes the new child of its grandparent by a *dchild CAS* we prove this node is not the marked node by contradiction by applying Lemma 5.19(5). Lemma 5.19(2) and

Lemma 5.19(3) are proved by using the induction hypothesis of each other. Furthermore, Lemma 5.19(4) can be proved using the fact that an *ichild CAS* always changes a pointer to a newly allocated node. Lemma 5.19(5) can be proved by applying Lemma 5.19(2) plus the fact that a *dchild CAS* changes the child pointer of $f.gpn$ from $f.pn$ to $f.other$. Thus, we can use Lemma 5.19(1) to show the contradiction that an unreachable node y becomes reachable in the execution shown in Figure 5.3, thereby correcting the flaw in the original proof of Lemma 14(7).

5.5 Proofs in the Backward Simulation

It is easier to prove the correctness of the backward simulation compared with the forward one, because our intention was to design the *IntAut* to be as similar to the *AbsAut* as possible. As discussed earlier, in Section 5.1, we were required to show that for each type of α such that $i' \xrightarrow{\alpha}_{IntAut} i$ and each $a \in states(AbsAut)$ such that $(i, a) \in bsr$, there exists a state a' of *AbsAut* and a sequence of actions $\hat{\alpha}$ such that $(i', a') \in bsr$ and $a' \overset{\hat{\alpha}}{\Longrightarrow}_{AbsAut} a$, and the external action in $\hat{\alpha}$ is the same as the external action in α.

Recall the backward simulation relation *bsr* and backward action correspondence defined in Section 4.3. It is trivial to prove that external actions, which are invocations and responses, satisfy the above properties. It is also not hard to prove that internal actions, except for $doInsertT(k, p)$ and $doDeleteT(k, p)$, satisfy this property, because they never modify shared objects which appear in the data relation part of bsr. A $doInsertT(k, p)$ action has a more complicated behaviour. It adds k into $i'.keys$, sets $i'.seen\_in(q)$ to be $true$ for any process q which is performing a $find(k)$ or an $insert(k)$ operation but has not decided to return a value, which allows us to linearize all $find(k)$ operations that subsequently return $true$ and all $insert(k)$ operations that subsequently return $false$ immediately after $doInsertT(k, p)$. For this action α, we need to construct the pre-state a' from a by removing k from $a.keys$. The value of program counter $a'.pc(q)$ for a process q is retrieved by setting its values to the precondition of q's action in $\hat{\alpha}$. Then, we show that $(i', a') \in bsr$ and $a' \stackrel{\hat{\alpha}}{\Longrightarrow}_{AbsAut} a$ and the external action in $\hat{\alpha}$ and α is the same. Because a $doDeleteT(k, p)$ action behaves in a symmetric way as $doInsertT(k, p)$, we used a similar method to construct $\hat{\alpha}$ and a' to complete the proof. Those proofs can all be found in our PVS scripts.

6 Conclusion

We believe that forward simulations are highly related to a concept called strong linearizability recently defined by Golab et al. [29]. We conjecture that an implementation is strongly linearizable if and only if there exists a forward simulation between the implementation and its sequential specification. Because we believe the BST algorithm is actually strongly linearizable, we believe that a forward simulation exists between the implementation and its sequential specification. Therefore, one may be tempted to try to prove that the concrete automaton (the implementation) implements the canonical automaton (specification) directly by a forward simulation. However, that relation is much more complicated to formalize. Even when we split the proof into a backward and forward simulation, the forward simulation is still complicated to be proved using PVS, because the pseudocode of the algorithm is far from trivial, the concrete automaton is complicated, and the program counter relationship defined in Figure 4.10 consists of a lot of possibilities. When we were proving a lemma about the concrete automaton, it was almost impossible to use PVS's built-in automated reasoning procedures such as "grind" to save time, because PVS "got lost" in those automatic generated subgoals when auto-rewriting the concrete automaton.

It seems more reasonable to use a forward simulation to prove the correctness of an implementation, when its pseudocode is short and simple, as Colvin et al. [16] and Doherty et al. [13] did. Another reasonable approach is to develop tools to automatically generate the I/O automaton model from the implementations, such as its program counter values and actions. The Tempo toolkits developed by Lynch et al. [20] can translate specifications described in an I/O automata like language into I/O automata and help with the verification. It would be easier to handle complicated implementations if we were to have a tool that automatically proves some easy facts about the pseudocode. For instance, local variables are not nil when they are used, and process p's local variables remains the same if p did not modify them. It may also be useful to have a tool to generate lemmas on the pseudocode structure, such as when a process is executing inside an IF block, the IF condition held at some earlier time. This would really save proofs designers' time and let them focus on the more important and difficult lemmas required to prove correctness. Lesani et al. [28] tried to construct a general framework for formally verifying software transactional memory algorithms. Their framework provides templates which make it easier to construct I/O automata and forward or backward relations.

We have formally verified the correctness of the simplified algorithm using PVS by showing that a forward simulation exists between the concrete automaton, which models the implementation, and the intermediate automaton, and a backward simulation exists between the intermediate automaton and the canonical one which models the sequential specification. Thus, the algorithm without the helping mechanism is linearizable. Our future work is to verify that the original algorithm with the helping mechanism is also linearizable. This can be done by building another concrete automaton and showing that there is a simulation between this newly built one and *ConcAut*. This may be applicable since the automata for the original algorithm and the simplified version are quite similar. An alternative way to verify the original implementation would be to model the algorithm as a concrete automaton and redo the backward and forward simulation proof again. In this approach, a lot of lemmas and proofs we have proved in the old automaton can be directly reused and that will save a great deal of time. When modelling the pseudocode as *ConcAut*, we considered that accessing unchangeable fields of a shared object can be considered as 0 step of accessing the shared memory. We planned to explore such property in the future, which may simplify the way of modelling of general concurrent pseudo codes, thereby simplifying a formal verification. The BST algorithm is a non-blocking algorithm, which means that it guarantees that in any infinite execution some operation completes. We are also interested in formalizing the proof of this progress property of the BST using PVS, which may be quite different from verifying the correctness property. In particular, it will require us to reason about infinite executions.

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