A Theory of Metric Labelled Transition Systems

Franck van Breugel*

McGill University
School of Computer Science
3480 University Street, Montreal H3A 2A7, Canada

Abstract

Labelled transition systems are useful for giving semantics to programming languages. Kok and Rutten have developed some theory to prove semantic models defined by means of labelled transition systems to be equal to other semantic models. Metric labelled transition systems are labelled transition systems with the configurations and actions endowed with metrics. The additional metric structure allows us to generalize the theory developed by Kok and Rutten.

Introduction

The classical result due to Banach [Ban22] that a *contractive function* from a nonempty complete *metric space* to itself has a *unique fixed point* plays an important role in the theory of *metric semantics* for programming languages. Metric spaces and Banach's theorem were first employed by Nivat [Niv79] to give semantics to recursive program schemes. Inspired by the work of Nivat, De Bakker and Zucker [BZ82] gave semantics to concurrent languages by means of metric spaces. The metric spaces they used were defined as solutions of recursive domain equations. By means of Banach's theorem America and Rutten [AR89] proved that a particular class of domain equations has unique solutions. Banach's theorem has also been used to prove semantic models to be equal. Kok and Rutten [KR90] applied a proof principle which we baptize the *unique fixed point proof principle*. By means of this proof principle elements of a metric space can be proved to be equal. First, one introduces a function from the metric space to itself. Second, one shows that the function is a contraction. And finally, one shows that the elements to be proved equal are fixed point of the contraction. To apply this proof principle to prove semantic models to be equal, the models should be element of a metric space. Furthermore, a contractive function from the metric space to itself with the semantic models as fixed point is needed. Kok and Rutten developed some theory to prove *operational* semantic models defined by means of *labelled transition systems* a la Plotkin [Plo81] equal to other semantic models—in particular denotational semantic models—by uniqueness of fixed point. For numerous applications of their theory we refer the reader to De Bakker and De Vink's textbook [BV95]. Their results are only applicable to operational semantic models induced by *finitely branching* labelled transition systems. Although most programming languages can be modelled operationally by means of a finitely branching labelled transition system, there are languages which cannot. For example, the real-time language ACPrρ introduced by Baeten and Bergstra in [BB91] gives rise to infinite branching.

In this paper we generalize the theory developed by Kok and Rutten. In the generalized setting we are able to deal with the above mentioned real-time language. To generalize the results we supply the labelled transition systems with some additional metric structure. These enriched labelled transition systems we call *metric labelled transition systems*. The additional metric structure enables us to generalize finitely branching to *compactly branching*. All results proved by Kok and Rutten for finitely branching labelled transition systems are generalized for compactly branching metric labelled transition systems. This amounts to a theory of metric labelled transition systems.

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1 Metric labelled transition systems

A metric labelled transition system is a labelled transition system with some additional structure. The structure is added by endowing the sets of configurations and actions with 1-bounded complete metrics.

Definition 1.1 A *metric labelled transition system* is a triple $(C, A, \rightarrow)$ consisting of

- a 1-bounded metric space of *configurations* C,

- a 1-bounded complete metric space of *actions* A, and

- a *transition relation* $\rightarrow \subseteq C \times A \times C$.

Instead of $(c, a, c') \in \rightarrow$ we write $c \xrightarrow{a} c'$. Most of the time we only present the transition relation of a metric labelled transition system. Frequently we depict (the transition relation of) a metric labelled transition system by a directed graph. The nodes are labelled with configurations and the edges are labelled with actions.
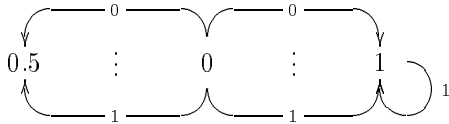
Example 1.2 The labelled transition system

$$(\{0, 0.5, 1\}, [0, 1], \{(0, a, 0.5) \mid a \in [0, 1]\} \cup \{(0, a, 1) \mid a \in [0, 1]\} \cup \{(1, 1, 1)\})$$

is presented by

$$\begin{cases} 0 \xrightarrow{a} 0.5 & \text{for } a \in [0, 1] \\ 0 \xrightarrow{a} 1 & \text{for } a \in [0, 1] \\ 1 \xrightarrow{1} 1 \end{cases}$$

and depicted by

By endowing the set of configurations and the set of actions both with the Euclidean metric we obtain a metric labelled transition system.

If $c \xrightarrow{a} c'$ then we say that there exists a transition from c to c' labelled with a. If there exists a transition from c then we call c a *nonterminal* configuration and write $c \rightarrow$. Otherwise we call c a *terminal* configuration and write $c \nrightarrow$.

Example 1.3 In Example 1.2 the configurations 0 and 1 are nonterminal and the configuration 0.5 is terminal.

A labelled transition system is called *finitely branching* if every configuration has only finitely many outgoing transitions. Because we have a metric on the sets of configurations and actions (and hence on the Cartesian product of these sets), finitely branching can be generalized to compactly branching: for each configuration, its set of outgoing transitions is compact.

Definition 1.4 A metric labelled transition system $(C, A, \rightarrow)$ is called *compactly branching* if, for all $c \in C$, the set

$$\mathcal{CB}(c) = \{(a, c') \mid c \xrightarrow{a} c'\}$$

is compact.

If we endow the configurations and the actions of a finitely branching labelled transition system both with an arbitrary 1-bounded (complete) metric, then we obtain a compactly branching metric labelled transition system. A compactly branching metric labelled transition system is in general not finitely branching.

Example 1.5 The metric labelled transition system introduced in Example 1.2 is not finitely branching but is compactly branching. If we endow the actions with the discrete metric then the obtained metric labelled transition system is no longer compactly branching.

For a compactly branching metric labelled transition system we introduce the condition of transitioning being nonexpansive. To formulate this condition we provide the compact sets of outgoing transitions of the configurations, elements of $\mathcal{P}_c\left(A \times C\right)$, with a metric. The action-configuration pairs are endowed with the metric obtained from the metric on the actions and the metric on the configurations multiplied by a $\frac{1}{2}$, denoted by $A \times \frac{1}{2} \cdot C$. As we will see below the introduction of the $\frac{1}{2}\cdot$ gives rise to a less restrictive condition. The (compact) sets of these pairs are endowed with the Hausdorff metric [Hau14].

Definition 1.6 A compactly branching metric labelled transition system $(C, A, \rightarrow)$ is called *nonexpansive* if the function

$$\mathcal{CB} : C \rightarrow \mathcal{P}_c\left(A \times \tfrac{1}{2} \cdot C\right)$$

defined by

$$\mathcal{CB}\left(c\right) = \{\, (a, c') \mid c \xrightarrow{a} c' \,\}$$
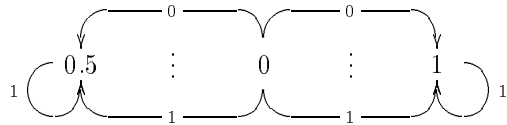
is nonexpansive.

Example 1.7 The compactly branching metric labelled transition system of Example 1.2 is not nonexpansive, because

$$
\begin{aligned}
d\left(\mathcal{CB}\left(0.5\right), \mathcal{CB}\left(1\right)\right) & \\
= \quad & d\left(\emptyset, \{(1, 1)\}\right) \\
= \quad & 1 \\
\not\leq \quad & 0.5 \\
= \quad & d\left(0.5, 1\right).
\end{aligned}
$$

By adding the transition

$$0.5 \xrightarrow{1} 0.5$$

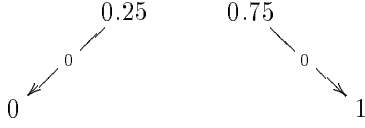we obtain the compactly branching metric labelled transition system

which is nonexpansive.

The $\frac{1}{2}\cdot$ in the above definition does not change the compactness condition. By leaving out the $\frac{1}{2}\cdot$ we obtain a more restrictive nonexpansiveness condition.

Example 1.8 The labelled transition system

$$
\begin{cases}
0.25 \xrightarrow{0} 0 \\
0.75 \xrightarrow{0} 1
\end{cases}
$$

depicted by

$$\begin{array}{ccc} & 0.25 \quad\quad\quad 0.75 & \\ {}_{0} & & {}_{0} \\ 0 & & 1 \end{array}$$

with the set of configurations endowed with the Euclidean metric, is (compactly branching and) nonexpansive, since

$$d\left(\mathcal{CB}\left(0.25\right),\mathcal{CB}\left(0.75\right)\right)$$
$$= \quad d\left(\{(0,0)\},\{(0,1)\}\right)$$
$$= \quad 0.5$$
$$= \quad d\left(0.25,0.75\right).$$

If we leave out the $\frac{1}{2}\cdot$ we have that

$$d\left(\mathcal{CB}\left(0.25\right),\mathcal{CB}\left(0.75\right)\right)$$
$$= \quad 1$$
$$\not\leq \quad 0.5$$
$$= \quad d\left(0.25,0.75\right).$$

A finitely branching labelled transition system with the configurations endowed with an arbitrary 1-bounded metric and the actions endowed with the discrete metric is (compactly branching and) nonexpansive. Consequently we have generalized finitely branching to compactly branching and nonexpansive.

2   Operational semantics

The operational semantics induced by a metric labelled transition system is a function assigning to each configuration a (nonempty) set of (finite and infinite) action sequences. This assignment is driven by the transition relation of the metric labelled transition system.

Definition 2.1 An *operational semantics* induced by a metric labelled transition system $(C, A, \rightarrow)$ is a function $\mathcal{O} : C \rightarrow \mathcal{P}_n\left(A^\infty\right)$ defined by

$$\mathcal{O}\left(c\right) = \{\, a_1 a_2 \cdots a_n \mid c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_n} c_{n+1} \not\rightarrow \,\}\cup$$
$$\{\, a_1 a_2 \cdots \quad \mid c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \,\}.$$

In the above definition we use

$$c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_n} c_{n+1} \not\rightarrow$$

as an abbreviation for

$$c = c_1 \land \forall 1 \leq m \leq n : c_m \xrightarrow{a_m} c_{m+1} \land c_{n+1} \not\rightarrow$$

and

$$c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots$$

as an abbreviation for

$$c = c_1 \land \forall m \geq 1 : c_m \xrightarrow{a_m} c_{m+1}.$$

A sequence $a_1 a_2 \cdots a_n$ is an element of the operational semantics of the configuration c if there exists a transition sequence from c to some terminal configuration labelled with $a_1 a_2 \cdots a_n$. If there exists an infinite transition sequence from c labelled with $a_1 a_2 \cdots$ then the infinite sequence $a_1 a_2 \cdots$ is an element of the operational semantics of c. Consequently the operational semantics of a terminal configuration is a singleton set consisting of the empty sequence ε. Note that each configuration is mapped to a nonempty set.

Example 2.2 The metric labelled transition system of Example 1.2 induces the operational semantics $\mathcal{O}$ defined by

$$\mathcal{O}\,(0) \;\;= [0,1] \cdot \{\,1^\omega\,\} \cup [0,1]$$
$$\mathcal{O}\,(0.5) = \{\varepsilon\}$$
$$\mathcal{O}\,(1) \;\;= \{\,1^\omega\,\}$$

To prove an operational semantics to be equal to another semantics by means of the unique fixed point proof principle, the operational semantics should be an element of a metric space. To turn the space $C \to \mathcal{P}_n\,(A^\infty)$ into a metric space we first endow the set of finite and infinite action sequences A^∞ with the following metric.

Definition 2.3 The metric $d_{A^\infty} : A^\infty \times A^\infty \to [0,1]$ is defined by

$$d_{A^\infty}\,(\sigma_1,\sigma_2) = \begin{cases} 0 & \text{if } \sigma_1 = \sigma_2 \\ \sup\left(\{\,2^{-n+1} \cdot d_A\,(\sigma_1\,(n),\sigma_2\,(n)) \mid 1 \le n \le |\sigma_1|\,\} \cup \{2^{-|\sigma_1|}\}\right) & \text{if } |\sigma_1| < |\sigma_2| \\ \sup\left(\{\,2^{-n+1} \cdot d_A\,(\sigma_1\,(n),\sigma_2\,(n)) \mid 1 \le n \le |\sigma_2|\,\} \cup \{2^{-|\sigma_2|}\}\right) & \text{if } |\sigma_1| > |\sigma_2| \\ \sup\{\,2^{-n+1} \cdot d_A\,(\sigma_1\,(n),\sigma_2\,(n)) \mid 1 \le n \le |\sigma_1|\,\} & \text{otherwise} \end{cases}$$

where $|\sigma_i|$ denotes the length of the sequence σ_i and $\sigma_i\,(n)$ denotes the n-th element of σ_i.

In case we endow the action set A with the discrete metric we obtain the usual Baire-like metric [Bai09]:

$$d_{A^\infty}\,(\sigma_1,\sigma_2) = \begin{cases} 0 & \text{if } \sigma_1 = \sigma_2 \\ 2^{-n} & \text{otherwise} \end{cases}$$

where n is the length of the longest common prefix of σ_1 and σ_2.

Second, we endow the (nonempty) sets of action sequences with the Hausdorff metric induced by the above introduced 1-bounded metric on action sequences. In this way we obtain a pseudometric space rather than a metric space. The restriction to (nonempty and) *compact* sets of action sequences gives rise to a metric space. Finally, the functions from configurations to (nonempty and compact) sets of action sequences are endowed with the supremum of the pointwise distances. We restrict our attention to operational semantic models being element of this metric space.

Definition 2.4 An operational semantics $\mathcal{O} : C \to \mathcal{P}_n\,(A^\infty)$ is called *compact* if $\mathcal{O} \in C \to \mathcal{P}_{nc}\,(A^\infty)$.

Example 2.5 The operational semantics presented in Example 2.2 is compact if the action set $[0,1]$ is endowed with the Euclidean metric. If we endow the action set $[0,1]$ with the discrete metric then the operational semantics is not compact any more.

Not every metric labelled transition system induces a compact operational semantics. If we restrict ourselves to compactly branching and nonexpansive metric labelled transition systems then we obtain compact operational semantic models. Without the additional nonexpansive condition we do in general not obtain compact operational semantic models.

Example 2.6 The compactly branching metric labelled transition system

$$\begin{cases} 0 \xrightarrow{\;0\;} 0 \\ 0 \xrightarrow{\;\frac{1}{n}\;} \frac{1}{n} \quad \text{for } n \in \mathbb{N} \end{cases}$$

depicted by
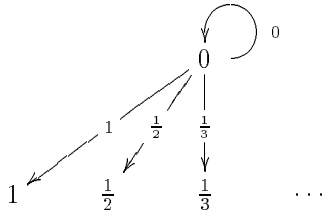
with the set of configurations and the set of actions both endowed with the Euclidean metric, does not induce a compact operational semantics. Note that the function $\mathcal{CB}$ is not nonexpansive.

Next we prove that a compactly branching and nonexpansive metric labelled transition system induces a compact operational semantics. To prove this we first prove two additional propositions. In the first proposition we demonstrate that the nonterminal and terminal configurations of a compactly branching and nonexpansive metric labelled transition system have distance 1 to each other.

Proposition 2.7 *The nonterminal and terminal configurations of a compactly branching and nonexpansive metric labelled transition system have distance 1 to each other.*

Proof For a nonterminal configuration c, $\mathcal{CB}(c) \neq \emptyset$ and for a terminal configuration c', $\mathcal{CB}(c') = \emptyset$. Since the metric labelled transition system is nonexpansive,

$$1 = d\left(\mathcal{CB}(c), \mathcal{CB}(c')\right) \leq d(c, c').$$

$\square$

In the second proposition we show that, for a compactly branching and nonexpansive metric labelled transition system, for all configurations c and natural numbers n, the set of transition sequences starting from the configuration c and truncated at length n is compact.

Proposition 2.8 *Let $(C, A, \rightarrow)$ be a compactly branching and nonexpansive metric labelled transition system. For all $c \in C$ and $n \in \mathbb{N}$, the set*

$$\mathcal{CB}^n(c) = \{\, (a_1, c_2, a_2, \ldots, a_n, c_{n+1}) \mid c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_n} c_{n+1} \,\}$$

is compact [1].

Proof This proposition is proved by induction on n. For $n = 0$ the proposition is vacuously true. Let $n > 0$. Let $c \in C$. Because the metric labelled transition system is compactly branching, for all $c_n \in C$, the set $\mathcal{CB}(c_n)$ is compact. Consequently, for all $c_2, \ldots, c_n \in C$ and $a_1, a_2, \ldots, a_{n-1} \in A$, the set

$$\{\, (a_1, c_2, a_2, \ldots, a_n, c_{n+1}) \mid c_n \xrightarrow{a_n} c_{n+1} \,\}$$

is also compact. Since the metric labelled transition system is nonexpansive, the function corresponding to the above set is nonexpansive in $(a_1, c_2, a_2, \ldots, a_{n-1}, c_n)$. By induction, the set $\mathcal{CB}^{n-1}(c)$ is compact. Because the nonexpansive image of a compact set is compact,

$$\left\{\{\, (a_1, c_2, a_2, \ldots, a_n, c_{n+1}) \mid c_n \xrightarrow{a_n} c_{n+1} \,\} \,\middle|\, (a_1, c_2, a_2, \ldots, a_{n-1}, c_n) \in \mathcal{CB}^{n-1}(c)\right\}$$

is a compact set of compact sets. From Michael's theorem (Theorem 2.5 of [Mic51]), the compactness of the set

$$\bigcup \left\{\{\, (a_1, c_2, a_2, \ldots, a_n, c_{n+1}) \mid c_n \xrightarrow{a_n} c_{n+1} \,\} \,\middle|\, (a_1, c_2, a_2, \ldots, a_{n-1}, c_n) \in \mathcal{CB}^{n-1}(c)\right\}$$

i.e. $\mathcal{CB}^n(c)$, can be concluded. $\square$

Now we are ready to prove the main result of this paper.

Theorem 2.9 *The operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is compact.*

Proof Let $(C, A, \rightarrow)$ be a compactly branching and nonexpansive metric labelled transition system. We prove that the induced operational semantics $\mathcal{O}$ is compact, i.e., for all $c \in C$, the set $\mathcal{O}(c)$ is compact.

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[1] To be precise, $\mathcal{CB}^n(c)$ is a compact subset of $A \times \frac{1}{2} \cdot (C \times A \times \frac{1}{2} \cdot (\cdots A \times \frac{1}{2} \cdot C))$. We leave it to the reader to fill in these details in the proof.

Let $c \in C$. Let $(\sigma_n)_n$ be a sequence in $\mathcal{O}(c)$. We show that there exists a subsequence $(\sigma_{s(n)})_n$ of $(\sigma_n)_n$ converging to some $\sigma \in \mathcal{O}(c)$.

The subsequence $(\sigma_{s(n)})_n$ will be constructed from a collection of subsequences $(\sigma_{s_m(n)})_n$ satisfying

$$(\forall m \in \mathbb{N} : Q(m)) \vee (\exists k \in \mathbb{N} : \forall 1 \leq m < k : Q(m) \wedge R(k)) \tag{1}$$

where

$$
\begin{aligned}
Q(m) \iff \forall n \in \mathbb{N} : \quad & \sigma_{s_m(n)} = a_{1,s_m(n)} a_{2,s_m(n)} \cdots a_{m,s_m(n)} \sigma_{m,s_m(n)} \wedge \\
& c = c_{1,s_m(n)} \xrightarrow{a_{1,s_m(n)}} c_{2,s_m(n)} \xrightarrow{a_{2,s_m(n)}} \cdots \xrightarrow{a_{m,s_m(n)}} c_{m+1,s_m(n)} \rightarrow \wedge \\
& \sigma_{m,s_m(n)} \in \mathcal{O}(c_{m+1,s_m(n)}) \wedge \\
& \forall 1 \leq j \leq m : \lim_h a_{j,s_m(h)} = a_j \wedge \\
& \forall 1 \leq j \leq m+1 : \lim_h c_{j,s_m(h)} = c_j \wedge \\
& c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_m} c_{m+1} \rightarrow
\end{aligned}
$$

and

$$
\begin{aligned}
R(m) \iff \forall n \in \mathbb{N} : \quad & \sigma_{s_m(n)} = a_{1,s_m(n)} a_{2,s_m(n)} \cdots a_{m,s_m(n)} \wedge \\
& c = c_{1,s_m(n)} \xrightarrow{a_{1,s_m(n)}} c_{2,s_m(n)} \xrightarrow{a_{2,s_m(n)}} \cdots \xrightarrow{a_{m,s_m(n)}} c_{m+1,s_m(n)} \nrightarrow \wedge \\
& \forall 1 \leq j \leq m : \lim_h a_{j,s_m(h)} = a_j \wedge \\
& \forall 1 \leq j \leq m+1 : \lim_h c_{j,s_m(h)} = c_j \wedge \\
& c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_m} c_{m+1} \nrightarrow .
\end{aligned}
$$

The existence of the subsequences $(\sigma_{s_m(n)})_n$ is verified by proving, for all $i \in \mathbb{N}$,

$$P(i) \iff (\forall 1 \leq m \leq i : Q(m)) \vee (\exists 1 \leq k \leq i : \forall 1 \leq m < k : Q(m) \wedge R(k))$$

by induction on i.

To prove $P(0)$ it suffices to show $Q(0) \vee R(0)$. Obviously the sequence $(\sigma_n)_n$ satisfies $Q(0) \vee R(0)$.

Let $i > 0$. To prove $P(i-1) \Rightarrow P(i)$ it suffices to show $Q(i-1) \Rightarrow Q(i) \vee R(i)$. If $Q(i-1)$ then

$$
\begin{aligned}
\forall n \in \mathbb{N} : \quad & ((\sigma_{s_{i-1}(n)} = a_{1,s_{i-1}(n)} a_{2,s_{i-1}(n)} \cdots a_{i,s_{i-1}(n)} \sigma_{i,s_{i-1}(n)} \wedge \\
& c = c_{1,s_{i-1}(n)} \xrightarrow{a_{1,s_{i-1}(n)}} c_{2,s_{i-1}(n)} \xrightarrow{a_{2,s_{i-1}(n)}} \cdots \xrightarrow{a_{i,s_{i-1}(n)}} c_{i+1,s_{i-1}(n)} \rightarrow \wedge \\
& \sigma_{i,s_{i-1}(n)} \in \mathcal{O}(c_{i+1,s_{i-1}(n)})) \vee \\
& (\sigma_{s_{i-1}(n)} = a_{1,s_{i-1}(n)} a_{2,s_{i-1}(n)} \cdots a_{i,s_{i-1}(n)} \wedge \\
& c = c_{1,s_{i-1}(n)} \xrightarrow{a_{1,s_{i-1}(n)}} c_{2,s_{i-1}(n)} \xrightarrow{a_{2,s_{i-1}(n)}} \cdots \xrightarrow{a_{i,s_{i-1}(n)}} c_{i+1,s_{i-1}(n)} \nrightarrow)) \wedge \\
& \forall 1 \leq j \leq i-1 : \lim_h a_{j,s_{i-1}(h)} = a_j \wedge \\
& \forall 1 \leq j \leq i : \lim_h c_{j,s_{i-1}(h)} = c_j \wedge \\
& c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{i-1}} c_i \rightarrow .
\end{aligned}
$$

Since the sequence

$$(a_{1,s_{i-1}(n)}, c_{2,s_{i-1}(n)}, a_{2,s_{i-1}(n)}, \ldots, a_{i,s_{i-1}(n)}, c_{i+1,s_{i-1}(n)})_i,$$

is a sequence in $\mathcal{CB}^i(c)$ and by Proposition 2.8 the set $\mathcal{CB}^i(c)$ is compact, the sequence has a subsequence

$$(a_{1,s'_{i-1}(n)}, c_{2,s'_{i-1}(n)}, a_{2,s'_{i-1}(n)}, \ldots, a_{i,s'_{i-1}(n)}, c_{i+1,s'_{i-1}(n)})_i$$

which converges to $(a_1, c_2, a_2, \ldots, a_i, c_{i+1})$ in $\mathcal{CB}^i(c)$ for some $a_i \in A$ and $c_{i+1} \in C$, i.e.

$$c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_i} c_{i+1}.$$

If $c_{i+1} \rightarrow$ (or $c_{i+1} \not\rightarrow$) then there exists a subsequence

$$\left(a_{1,s_i(n)}, c_{2,s_i(n)}, a_{2,s_i(n)}, \ldots, a_{k,s_i(n)}, c_{i+1,s_i(n)}\right)_i$$

of the sequence

$$\left(a_{1,s'_{i-1}(n)}, c_{2,s'_{i-1}(n)}, a_{2,s'_{i-1}(n)}, \ldots, a_{i,s'_{i-1}(n)}, c_{i+1,s'_{i-1}(n)}\right)_i$$

satisfying $c_{i+1,s_i(n)} \rightarrow$ (or $c_{i,s_i(n)} \not\rightarrow$), since the nonterminal and terminal configurations have distance 1 to each other according to Proposition 2.7. Consequently $Q(i)$ (or $R(i)$).

From the subsequences $(\sigma_{s_{m(n)}})_n$ satisfying (1) we construct the subsequence $(\sigma_{s(n)})_n$ distinguishing the following two cases.

1. If $\forall m \in \mathbb{N} : Q(m)$ then we define $s(n) = s_n(n)$. In this case, the sequence $(\sigma_{s(n)})_n$ converges to $\sigma = a_1 a_2 \cdots$ in $\mathcal{O}(c)$.

2. If $\exists k \in \mathbb{N} : \forall 1 \le m < k : Q(m) \land R(k)$ then we define $s = s_k$. The sequence $(\sigma_{s(n)})_n$ converges to $\sigma = a_1 a_2 \cdots a_k$ in $\mathcal{O}(c)$.

$\square$

Given a finitely branching labelled transition system $(C, A, \rightarrow)$, we endow the action set A with the discrete metric (consequently, the metric on A^∞ is the usual Baire-like metric) and the configuration set C with an arbitrary 1-bounded metric. We obtain a compactly branching and nonexpansive metric labelled transition system. According to the above theorem the corresponding operational semantics is compact. Hence, the folklore result that a finitely branching labelled transition system induces a compact operational semantics is a consequence of the above theorem.

The operational semantics induced by a compactly branching and nonexpansive metric labelled transition system has another property besides being compact: it is nonexpansive. The nonexpansiveness of a compact operational semantics is crucial when we want to apply the unique fixed point proof principle (the details will be supplied in Section 3).

Theorem 2.10 *The compact operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is nonexpansive.*

Proof Let $(C, A, \rightarrow)$ be a compactly branching and nonexpansive metric labelled transition system. Let $\mathcal{O}$ be the induced compact operational semantics. To prove the nonexpansiveness of $\mathcal{O}$, a sequence $(\mathcal{O}_n)_n$ of nonexpansive functions converging to $\mathcal{O}$ is introduced. Because the set of nonexpansive functions $C \rightarrow^1 \mathcal{P}_{nc}(A^\infty)$ is closed (a consequence of the completeness of A and Lemma 3 of Kuratowski's [Kur56]), we can conclude that $\mathcal{O}$ is nonexpansive. The function $\mathcal{O}_n : C \rightarrow \mathcal{P}_n(A^\infty)$ is defined by

$$\mathcal{O}_n(c) = \{ a_1 a_2 \cdots a_{k-1} \mid c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{k-1}} c_k \not\rightarrow \land k \le n+1 \} \cup$$
$$\{ a_1 a_2 \cdots a_{n-1} \mid c = c_1 \xrightarrow{a_1} c_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} c_n \rightarrow \}.$$

We have left to prove that, for all n, $\mathcal{O}_n \in C \rightarrow^1 \mathcal{P}_{nc}(A^\infty)$. We prove this by induction on n. Obviously, $\mathcal{O}_0 \in C \rightarrow^1 \mathcal{P}_{nc}(A^\infty)$. Assume $n > 0$. Let $c \in C$. By definition,

$$\mathcal{O}_n(c) = \begin{cases} \{\varepsilon\} & \text{if } c \not\rightarrow \\ \{ a\sigma \mid c \xrightarrow{a} c' \land \sigma \in \mathcal{O}_{n-1}(c') \} & \text{otherwise} \end{cases}$$

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Clearly, the set $\mathcal{O}_n(c)$ is nonempty.

Next, we show that the set $\mathcal{O}_n(c)$ is compact. By induction, $\mathcal{O}_{n-1}$ delivers compact sets. One can easily verify that, for all $a \in A$ and $c' \in C$, the set

$$\{\, a\sigma \mid \sigma \in \mathcal{O}_{n-1}(c') \,\}$$

is compact. By induction, $\mathcal{O}_{n-1}$ is nonexpansive. As a consequence, the function corresponding to the above set is nonexpansive in a and c'. Because the metric labelled transition system is compactly branching and the nonexpansive image of a compact set is compact,

$$\left\{\, \{\, a\sigma \mid \sigma \in \mathcal{O}_{n-1}(c') \,\} \;\middle|\; c \xrightarrow{a} c' \right\}$$

is a compact set of compact sets. According to Michael's theorem, the set

$$\bigcup \left\{\, \{\, a\sigma \mid \sigma \in \mathcal{O}_{n-1}(c') \,\} \;\middle|\; c \xrightarrow{a} c' \right\},$$

i.e. $\mathcal{O}_n(c)$ is compact. Also $\{\varepsilon\}$ is a compact set. Hence, the set $\mathcal{O}_n(c)$ is compact.

Finally, the nonexpansiveness of $\mathcal{O}_n$ is shown. We have to show that, for all c_1, $c_2 \in C$,

$$d(\mathcal{O}_n(c_1), \mathcal{O}_n(c_2)) \le d(c_1, c_2).$$

If both c_1 and c_2 are terminal configurations then the above is vacuously true. Because the nonterminal and terminal configurations have distance 1 to each other (Proposition 2.7), the above is also true if one of the configurations is a nonterminal configuration and the other one is a terminal configuration. That leaves us only the case that both c_1 and c_2 are nonterminal configurations. In that case,

$$d(\mathcal{O}_n(c_1), \mathcal{O}_n(c_2))$$
$$= \ d(\{\, a_1\sigma_1 \mid c_1 \xrightarrow{a_1} c_1' \wedge \sigma_1 \in \mathcal{O}_{n-1}(c_1') \,\}, \{\, a_2\sigma_2 \mid c_2 \xrightarrow{a_2} c_2' \wedge \sigma_2 \in \mathcal{O}_{n-1}(c_2') \,\})$$
$$= \ d\left(\bigcup \left\{\, \{\, a_1\sigma_1 \mid \sigma_1 \in \mathcal{O}_{n-1}(c_1') \,\} \;\middle|\; c_1 \xrightarrow{a_1} c_1' \right\}, \bigcup \left\{\, \{\, a_2\sigma_2 \mid \sigma_2 \in \mathcal{O}_{n-1}(c_2') \,\} \;\middle|\; c_2 \xrightarrow{a_2} c_2' \right\}\right)$$
$$\le \ d\left(\left\{\, \{\, a_1\sigma_1 \mid \sigma_1 \in \mathcal{O}_{n-1}(c_1') \,\} \;\middle|\; c_1 \xrightarrow{a_1} c_1' \right\}, \left\{\, \{\, a_2\sigma_2 \mid \sigma_2 \in \mathcal{O}_{n-1}(c_2') \,\} \;\middle|\; c_2 \xrightarrow{a_2} c_2' \right\}\right)$$
$$[\bigcup \text{ is nonexpansive}]$$
$$\le \ d(\{\, (a_1, \mathcal{O}_{n-1}(c_1')) \mid c_1 \xrightarrow{a_1} c_1' \,\}, \{\, (a_2, \mathcal{O}_{n-1}(c_2')) \mid c_2 \xrightarrow{a_2} c_2' \,\})$$
$$\le \ d(\{\, (a_1, c_1') \mid c_1 \xrightarrow{a_1} c_1' \,\}, \{\, (a_2, c_2') \mid c_2 \xrightarrow{a_2} c_2' \,\}) \quad [\text{by induction, } \mathcal{O}_{n-1} \text{ is nonexpansive}]$$
$$\le \ d(c_1, c_2) \quad [\text{the metric labelled transition system is nonexpansive}]$$

$\square$

3 Semantics transformations

In the previous section we have shown that the operational semantics induced by a compactly branching and nonexpansive metric labelled transition system is an element of a metric space. To apply the unique fixed point proof principle we have to introduce a contractive function from the metric space to itself with the operational semantics as fixed point. We call this function a semantics transformation: a function transforming a semantics into another semantics. Like an operational semantics, a semantics transformation is induced by a metric labelled transition system.

Definition 3.1 A *semantics transformation* induced by a metric labelled transition system $(C, A, \rightarrow)$ is a function

$$\mathfrak{T} : (C \rightarrow \mathcal{P}_n(A^\infty)) \rightarrow (C \rightarrow \mathcal{P}_n(A^\infty))$$

defined by

$$\mathfrak{T}(\mathcal{S})(c) = \begin{cases} \{\varepsilon\} & \text{if } c \not\rightarrow \\ \{\, a\sigma \mid c \xrightarrow{a} c' \wedge \sigma \in \mathcal{S}(c) \,\} & \text{otherwise} \end{cases}$$

The semantics transformation $\mathfrak{T}$ transforms the semantics $\mathcal{S}$ into the semantics $\mathfrak{T}(\mathcal{S})$. This semantics $\mathfrak{T}(\mathcal{S})$ assigns to a terminal configuration the singleton set consisting of the empty sequence ε. To a non-terminal configuration c, the semantics $\mathfrak{T}(\mathcal{S})$ assigns the set of sequences $a\sigma$ obtained from the label a of a transition (of the metric labelled transition system inducing the semantics transformation) from the nonterminal configuration c to some configuration c', and a sequence σ of $\mathcal{S}(c')$.

Proposition 3.2 *The operational semantics $\mathcal{O}$ induced by a metric labelled transition system is a fixed point of the semantics transformation $\mathfrak{T}$ induced by the metric labelled transition system, i.e.*

$$\mathcal{O} = \mathfrak{T}(\mathcal{O}).$$

Proof Let $\mathcal{O}$ and $\mathfrak{T}$ be the operational semantics and the semantics transformation induced by the metric labelled transition system $(C, A, \rightarrow)$. Let $c \in C$. Obviously $\mathfrak{T}(\mathcal{O})(c) = \mathcal{O}(c)$ if $c \not\rightarrow$. Otherwise, for all $\sigma \in \mathcal{P}_n(A^\infty)$,

$$\sigma \in \mathfrak{T}(\mathcal{O})(c)$$
$$\Longleftrightarrow \quad \exists a \in A : \exists \sigma' \in \mathcal{P}_n(A^\infty) : \exists c' \in C : \sigma = a\sigma' \wedge c \xrightarrow{a} c' \wedge \sigma' \in \mathcal{O}(c')$$
$$\Longleftrightarrow \quad \sigma \in \mathcal{O}(c).$$

$\square$

According to the above proposition a semantics transformation has a fixed point. This fixed point is not necessarily unique.

Example 3.3 Consider the semantics transformation $\mathfrak{T}$ induced by the metric labelled transition system of Example 2.6. According to Proposition 3.2, the corresponding operational semantics $\mathcal{O}$ given by

$$\mathcal{O}(0) = \{\, 0^m \tfrac{1}{n} \mid m, n \in \mathbb{N} \,\} \cup \{0^\omega\}$$
$$\mathcal{O}(\tfrac{1}{n}) = \{\varepsilon\} \qquad\qquad\qquad \text{for } n \in \mathbb{N}$$

is a fixed point of $\mathfrak{T}$. Also the semantics $\mathcal{S}$ defined by

$$\mathcal{S}(0) = \{\, 0^m \tfrac{1}{n} \mid m, n \in \mathbb{N} \,\}$$
$$\mathcal{S}(\tfrac{1}{n}) = \{\varepsilon\} \qquad\qquad\qquad \text{for } n \in \mathbb{N}$$

is a fixed point of $\mathfrak{T}$.

We restrict ourselves to semantic transformations transforming compact and nonexpansive semantics into compact and nonexpansive semantics.

Definition 3.4 A semantics transformation

$$\mathfrak{T} : (C \rightarrow \mathcal{P}_n(A^\infty)) \rightarrow (C \rightarrow \mathcal{P}_n(A^\infty))$$

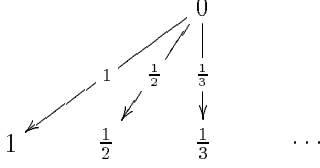is called *compactness and nonexpansiveness preserving* if

$$\mathfrak{T} \in (C \rightarrow^1 \mathcal{P}_{nc}(A^\infty)) \rightarrow (C \rightarrow^1 \mathcal{P}_{nc}(A^\infty)).$$

Not every metric labelled transition system induces a compactness and nonexpansiveness preserving semantics transformation.

Example 3.5 Consider the metric labelled transition system

$$0 \xrightarrow{\frac{1}{n}} \tfrac{1}{n} \text{ for } n \in \mathbb{N}$$

depicted by

with the set of configurations and the set of actions endowed with the Euclidean metric. Although the semantics $\mathcal{S}$, defined by, for all c,

$$\mathcal{S}(c) = \{\varepsilon\}$$

is compact, the semantics $\mathfrak{T}(\mathcal{S})$ is not compact.

Not even a compactly branching metric labelled transition system induces a compactness and nonexpansiveness preserving semantics transformation.

Example 3.6 The metric labelled transition system of Example 2.6 is compactly branching. The semantics $\mathcal{S}$, defined by, for all c,

$$\mathcal{S}(c) = \{\varepsilon\}$$

is compact and nonexpansive. The semantics $\mathfrak{T}(\mathcal{S})$ is compact but not nonexpansive.

But a compactly branching and nonexpansive metric labelled transition system gives rise to a compactness and nonexpansiveness preserving semantics transformation.

Theorem 3.7 *The semantics transformation induced by a compactly branching and nonexpansive metric labelled transition system is compactness and nonexpansiveness preserving.*

Proof Similar to the induction step of the proof of Theorem 2.10. $\qquad\square$

A compactness and nonexpansiveness preserving semantics transformation is a function from a metric space to itself. According to Proposition 3.2 and Theorem 2.9 and 2.10, the corresponding operational semantics is a fixed point of the semantics transformation. To be able to apply the unique fixed point proof principle we have left to prove that the semantics transformation is contractive.

Proposition 3.8 *A compactness and nonexpansiveness preserving semantics transformation is contractive.*

Proof Let $\mathfrak{T} : (C \to^1 \mathcal{P}_{nc}(A^\infty)) \to (C \to^1 \mathcal{P}_{nc}(A^\infty))$ be a compactness and nonexpansiveness preserving semantics transformation. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in C \to^1 \mathcal{P}_{nc}(A^\infty)$ and $c \in C$. We show that

$$d(\mathfrak{T}(\mathcal{S}_1)(c), \mathfrak{T}(\mathcal{S}_2)(c)) \leq \tfrac{1}{2} \cdot d(\mathcal{S}_1, \mathcal{S}_2).$$

We distinguish two cases.

1. If $c \not\to$, then
$$
\begin{aligned}
d(\mathfrak{T}&(\mathcal{S}_1)(c), \mathfrak{T}(\mathcal{S}_2)(c)) \\
&= \ d(\{\varepsilon\}, \{\varepsilon\}) \\
&\leq \ \tfrac{1}{2} \cdot d(\mathcal{S}_1, \mathcal{S}_2).
\end{aligned}
$$

2. If $c \rightarrow$, then

$$d\left(\mathfrak{T}(\mathcal{S}_1)(c), \mathfrak{T}(\mathcal{S}_2)(c)\right)$$
$$= d\left(\bigcup\left\{\{\, a\sigma_1 \mid \sigma_1 \in \mathcal{S}_1(c')\,\} \mid c \xrightarrow{a} c'\right\}, \bigcup\left\{\{\, a\sigma_2 \mid \sigma_2 \in \mathcal{S}_2(c')\,\} \mid c \xrightarrow{a} c'\right\}\right)$$
$$\leq d\left(\left\{\{\, a\sigma_1 \mid \sigma_1 \in \mathcal{S}_1(c')\,\} \mid c \xrightarrow{a} c'\right\}, \left\{\{\, a\sigma_2 \mid \sigma_2 \in \mathcal{S}_2(c')\,\} \mid c \xrightarrow{a} c'\right\}\right)$$
$$[\bigcup \text{ is nonexpansive}]$$
$$\leq \sup\{\, d\left(\{\, a\sigma_1 \mid \sigma_1 \in \mathcal{S}_1(c')\,\}, \{\, a\sigma_2 \mid \sigma_2 \in \mathcal{S}_2(c')\,\}\right) \mid c \xrightarrow{a} c'\,\}$$
$$= \sup\{\, \tfrac{1}{2} \cdot d\left(\mathcal{S}_1(c'), \mathcal{S}_2(c')\right) \mid c \xrightarrow{a} c'\,\}$$
$$\leq \tfrac{1}{2} \cdot d\left(\mathcal{S}_1, \mathcal{S}_2\right).$$

$\square$

Combining the above results we arrive at

Theorem 3.9 *The operational semantics $\mathcal{O}$ induced by a compactly branching and nonexpansive metric labelled transition system is the unique fixed point of the semantics transformation $\mathfrak{T}$ induced by the metric labelled transition system, i.e.*

$$\mathcal{O} = \text{fix}\,(\mathfrak{T}).$$

The above theorem generalizes the result of Kok and Rutten that the operational semantics induced by a finitely branching labelled transition system is the unique fixed point of the semantics transformation induced by the labelled transition system.

We conclude this section with an example motivating our restriction to (compact and) nonexpansive semantics.

Example 3.10 The metric labelled transition system

$$c \xrightarrow{a} c' \text{ for } c, c' \in [0, 1] \text{ and } a \in [0, 1]$$

with the configurations and the actions endowed with the Euclidean metric, is compactly branching and nonexpansive. Given the compact semantics $\mathcal{S}$ defined by

$$\mathcal{S}(c) = \left\{ \begin{array}{ll} \{1^n\} & \text{if } c = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ \{\varepsilon\} & \text{otherwise} \end{array} \right.$$

the semantics $\mathfrak{T}(\mathcal{S})$ is not compact.

Conclusion

Already in the early sixties the problem what structure to add to an abstract machine—like a labelled transition system—to obtain a topological machine was formulated by Ginsburg [Gin62]. Shreider [Shr64] introduced a particular topological machine—a compact automaton—to study dynamic programming. A general and detailed study of topological machines was developed by Brauer [Bra70]. Our metric labelled transition systems are a special instance of his topological machines. Another instance are Kent's metrical transition systems [Ken87]. A metrical transition system is a labelled transition system with the configurations endowed with a (generalized) ultrametric (the labels are not provided with any additional structure). Neither Brauer nor Kent uses the topological machines to give (operational) semantics as we have done in this paper.

This paper only describes part of a theory of metric labelled transition systems developed in the author's thesis [Bre94b]. We have shown how to generalize finitely branching to compactly branching and nonexpansive. Similarly *image finite* can be generalized to *image compact* and *binonexpansive*. The semantic models we have considered in this paper are *linear*: they assign to each configuration a set of sequences. By means of (metric) labelled transition systems one can also define *branching* operational semantic models. These

semantic models assign to each configuration a tree-like object. The details are provided in [Bre94b]. For an overview we refer the reader to [Bre94a].

If we replace nonexpansiveness by continuity (in Definition 1.6) all results still hold. In that case the induced operational semantic models are continuous (and not nonexpansive in general). In several applications of the theory the nonexpansiveness of the operational semantics is crucial (e.g. in application 2. below).

The presented theory has been applied in semantics. We mention three applications.

1. An operational and a denotational semantics for a fragment of the real-time language ACPrρ have been proved to be equal by uniqueness of fixed point exploiting the theory in [Bre91].

2. The theory has also been used to relate an operational and a denotational semantics for a higher-order language in [BB93].

3. Metric labelled transition systems have turned out to be very convenient to define abstraction operators as has been shown in [Bre95].

We are interested to see whether a similar theory can be developed if we replace the metric spaces of a metric labelled transition system by *algebraic complete generalized* metric spaces. To develop a theory of generalized metric labelled transition systems we have to restrict ourselves to nonexpansive and continuous semantics (for generalized metric spaces nonexpansiveness does not imply continuity as has been shown in, e.g., [Rut95]) rather than nonexpansive semantics, and instead of using the hyperspace of nonempty and compact sets endowed with the Hausdorff metric we have to employ the generalized convex powerdomain as defined in [BBR95].

Acknowledgements

References

[AR89]   P. America and J.J.M.M. Rutten. Solving Reflexive Domain Equations in a Category of Complete Metric Spaces. *Journal of Computer and System Sciences*, 39(3):343–375, December 1989.

[Bai09]   R. Baire. Sur la Représentation des Fonctions Discontinues. *Acta Mathematica*, 32(1):97–176, 1909.

[Ban22]   S. Banach. Sur les Opérations dans les Ensembles Abstraits et leurs Applications aux Equations Intégrales. *Fundamenta Mathematicae*, 3:133–181, 1922.

[BB91]   J.C.M. Baeten and J.A. Bergstra. Real Time Process Algebra. *Formal Aspects of Computing*, 3(2):142–188, April/June 1991.

[BB93]   J.W. de Bakker and F. van Breugel. Topological Models for Higher Order Control Flow. In S. Brookes, M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of the 9th International Conference on Mathematical Foundations of Programming Semantics*, volume 802 of *Lecture Notes in Computer Science*, pages 122–142, New Orleans, April 1993. Springer-Verlag.

[BBR95]   M.M. Bonsangue, F. van Breugel, and J.J.M.M. Rutten. Generalized Ultrametric Spaces: Completion, Topology, and Powerdomains via the Yoneda Embedding. Report CS-R9560, CWI, Amsterdam, September 1995.

[Bra70]   W. Brauer. *Zu den Grundlagen einer Theorie Topologischer Sequentieller Systeme und Automaten*, volume 31 of *Berichte der Gesellschaft für Mathematik und Datenverarbeitung*. Gesellschaft für Mathematik und Datenverarbeitung, Bonn, June 1970.

[Bre91]   F. van Breugel. Comparative Semantics for a Real-Time Programming Language with Integration. In S. Abramsky and T.S.E. Maibaum, editors, *Proceedings of the 4th International Conference on Theory and Practice of Software Development*, volume 493 of *Lecture Notes in Computer Science*, pages 397–411, Brighton, April 1991. Springer-Verlag.

[Bre94a] F. van Breugel. Generalizing Finiteness Conditions of Labelled Transition Systems. In S. Abiteboul and E. Shamir, editors, *Proceedings of the 21th International Colloquium on Automata, Languages, and Programming*, volume 820 of *Lecture Notes in Computer Science*, pages 376–387, Jerusalem, July 1994. Springer-Verlag.

[Bre94b] F. van Breugel. *Topological Models in Comparative Semantics*. PhD thesis, Vrije Universiteit, Amsterdam, September 1994.

[Bre95]   F. van Breugel. From Branching to Linear Metric Domains (and back). Report RS-95-30, BRICS, Aarhus, June 1995.

[BV95]   J.W. de Bakker and E.P. de Vink. *Control Flow Semantics*. Foundations of Computing Series. The MIT Press, Cambridge, 1995. To appear.

[BZ82]   J.W. de Bakker and J.I. Zucker. Processes and the Denotational Semantics of Concurrency. *Information and Control*, 54(1/2):70–120, July/August 1982.

[Gin62]   S. Ginsburg. *An Introduction to Mathematical Machine Theory*. Series in Computer Science and Information Processing. Addison-Wesley, Reading, Massachusetts, 1962.

[Hau14]   F. Hausdorff. *Grundzüge der Mengenlehre*. Leipzig, 1914.

[Ken87]   R.E. Kent. The Metric Closure Powerspace Construction. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of the 3rd Workshop on Mathematical Foundations of Programming Language Semantics*, volume 298 of *Lecture Notes in Computer Science*, pages 173–199, New Orleans, April 1987. Springer-Verlag.

[KR90]   J.N. Kok and J.J.M.M. Rutten. Contractions in Comparing Concurrency Semantics. *Theoretical Computer Science*, 76(2/3):179–222, November 1990.

[Kur56]   K. Kuratowski. Sur une Méthode de Métrisation Complète des Certains Espaces d'Ensembles Compacts. *Fundamenta Mathematicae*, 43(1):114–138, 1956.

[Mic51]   E. Michael. Topologies on Spaces of Subsets. *Transactions of the American Mathematical Society*, 71(1):152–182, July 1951.

[Niv79]   M. Nivat. Infinite Words, Infinite Trees, Infinite Computations. In J.W. de Bakker and J. van Leeuwen, editors, *Foundations of Computer Science III*, part 2: Languages, Logic, Semantics, volume 109 of *Mathematical Centre Tracts*, pages 3–52. Mathematical Centre, Amsterdam, 1979.

[Plo81]   G.D. Plotkin. A Structural Approach to Operational Semantics. Report DAIMI FN-19, Aarhus University, Aarhus, September 1981.

[Rut95]   J.J.M.M. Rutten. Elements of Generalized Ultrametric Domain Theory. Report CS-R9507, CWI, Amsterdam, January 1995.

[Shr64]   Yu.A. Shreider. Automata and the Problem of Dynamic Programming. *Problems of Cybernetics*, 5(1):33–58, October 1964.