Alexandroff and Scott Topologies for Generalized Ultrametric Spaces

M.M. Bonsangue,* F. van Breugel,† and J.J.M.M. Rutten‡

Abstract

Both preorders and ordinary ultrametric spaces are instances of generalized ultrametric spaces. Every generalized ultrametric space can be isometrically embedded in a (complete) function space by means of an ultrametric version of the categorical *Yoneda Lemma*. This simple fact gives naturally rise to: 1. a topology for generalized ultrametric spaces which for arbitrary preorders corresponds to the Alexandroff topology and for ordinary ultrametric spaces reduces to the ϵ-ball topology; 2. a topology for algebraic complete generalized ultrametric spaces generalizing both the Scott topology for arbitrary algebraic complete partial orders and the ϵ-ball topology for complete ultrametric spaces.

1 Introduction

Partial orders and metric spaces play a central role in the semantics of programming languages (cf., e.g., the recent textbooks [Win93] and [BV95]). Parts of their theory have been developed because of semantic necessity (see, e.g., [SP82] and [AR89]). Generalized ultrametric spaces provide a common framework for the study of both preorders and ordinary ultrametric spaces. A generalized ultrametric space consists of a set X together with a distance function $X(-,-) : X \times X \to [0,1]$ satisfying $X(x,x) = 0$ and $X(x,z) \leq \max\{X(x,y), X(y,z)\}$ for all x, y and z in X. The family of generalized ultrametric spaces contains all ordinary ultrametric spaces as well as all preordered spaces.

Generalized metric spaces were introduced by Lawvere [Law73] as an illustration of the thesis that fundamental structures are categories. The present work is inspired by Lawvere's enriched-categorical view of generalized metric spaces [Law73] as well as the more topological view of Smyth on quasi metric spaces [Smy87, Smy91]. It is based on the work [Rut95], in which some of the basic theory of generalized ultrametric spaces has

*Department of Computer Science, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands, email: `marcello@cs.vu.nl`.

†School of Computer Science, McGill University, 3480 University Street, Montreal H3A 2A7, Canada, email: `franck@cs.mcgill.ca`. Supported by the Netherlands Organization for Scientific Research.

‡CWI, P.O. Box 94079, 1090 AB Amsterdam, The Netherlands, email: `jan.rutten@cwi.nl`.

been developed, and is part of [BBR95], in which also completion and powerdomains for generalized ultrametric spaces are studied by using the Yoneda embedding.

We propose two topologies on generalized ultrametric spaces. The first topology is a generalized Alexandroff topology: for preorders this topology coincides with the Alexandroff topology while for ultrametric spaces it corresponds to the ϵ-ball topology. The second topology is a generalized Scott topology: for algebraic complete partial orders this topology corresponds to the Scott topology while for complete ultrametric spaces it coincides with the ϵ-ball topology. Both topologies are defined in two ways: by giving the open sets and by a closure operator. For both topologies the two alternative definitions are shown to coincide.

Our definition of the generalized Alexandroff topology in terms of open sets is similar to the ones given by Smyth [Smy87, Smy91] and Flagg and Kopperman [FK95]. A definition of a generalized Scott topology in terms of open sets similar to ours is briefly mentioned by Smyth in [Smy87]. The definitions of the topologies in terms of closure operators are new. The key observation—first made by Lawvere [Law73, Law86]—is that, intuitively, one may identify elements x of a generalized ultrametric space X with a description of the distances between any element y in X and x. Formally, this description is a function mapping every y in X to the distance $X(y,x)$. These functions can be interpreted as 'fuzzy' predicates on X: the value a function ϕ assigns to an element y in X is thought of as a measure for 'the extent to which y is an element of ϕ'. This observation corresponds to a generalized ultrametric version of the categorical Yoneda Lemma [Yon54]. The corresponding Yoneda embedding isometrically embeds a generalized ultrametric space X into the generalized ultrametric space of fuzzy predicates on X. By comparing the fuzzy predicates on X with the subsets of X we obtain the closure operator defining the generalized Alexandroff topology. Similarly, an algebraic complete generalized ultrametric space X can be embedded isometrically and continuously into the generalized ultrametric space of fuzzy predicates on its base B. By comparing the fuzzy predicates on B with the subsets of X we obtain the closure operator defining the generalized Scott topology.

The results presented here on generalized ultrametric spaces apply as well to generalized metric spaces (where one would have $X(x,z) \leq X(x,y) + X(y,x)$ for all x, y and z in X). We considered generalized ultrametric space because the distance induced by a preorder is indeed a generalized ultrametric. Moreover, because of the strong triangle inequality, ultrametric space are—from a computational point of view—better behaved than metric spaces and seem to arise naturally in the semantics of programming languages (cf. [Rut95]).

The paper is organized as follows. Sections 2 and 4 give the basic definitions and facts on generalized ultrametric spaces. The Yoneda Lemma and the generalized Alexandroff topology are discussed in Section 3, while the generalized Scott topology is presented in Section 5. Finally Section 6 discusses related work.

Acknowledgments: The authors are grateful to Jaco de Bakker, Paul Gastin, Pietro di Giannantonio, Bart Jacobs, Maurice Nivat, Bill Rounds, Erik de Vink and Kim Wagner for suggestions, comments and discussions. We wish to thank Bob Flagg and Philippe Sünderhauf for their careful reading of an earlier version of the paper: Lemma 5.5 is due to them as well as an improvement of Propositions 5.7, 5.8 and 5.9.

2 Generalized ultrametric spaces

A *generalized ultrametric space* (gum, for short) is a set X together with a distance function $X(-,-) : X \times X \to [0,1]$, which satisfies, for all x, y, and z in X,

1. $X(x,x) = 0$, and
2. $X(x,z) \leq \max\{X(x,y), X(y,z)\}$.

An example of a gum is the set of real numbers $[0,1]$ with distance function $[0,1](-,-) : [0,1] \times [0,1] \to [0,1]$ defined, for r and s in $[0,1]$, by

$$[0,1](r,s) = \begin{cases} 0 & \text{if } r \geq s \\ s & \text{if } r < s. \end{cases}$$

The gum $[0,1]$ has the following fundamental property.

Proposition 2.1 *For all $r, s, t \in [0,1]$, $\max\{s,t\} \geq r$ if and only if $s \geq [0,1](t,r)$.* $\square$

The above proposition expresses that the category with elements in $[0,1]$ as objects and the relation $\geq$ defining the morphisms between objects is cartesian closed. This category is also monoidal closed. Many properties of gum's derive from this categorical structure on $[0,1]$ (cf. [Law73, Rut95, BBR95]).

A gum generally does not satisfy

3. if $X(x,y) = 0$ and $X(y,x) = 0$ then $x = y$,
4. $X(x,y) = X(y,x)$,

which are the additional conditions that hold for an *ordinary* ultrametric space. Therefore it is sometimes called a *pseudo-quasi* ultrametric space. A *quasi* ultrametric space is a gum which furthermore satisfies axiom 3. A gum satisfying also axiom 4. is called a *pseudo* ultrametric space.

A map $f : X \to Y$ between gum's X and Y is *non-expansive* if for all x and x' in X,

$$Y(f(x), f(x')) \leq X(x,x').$$

The map f is called *isometric* if the above inequality is in fact an equality. We denote by Y^X the set of all non-expansive maps from X to Y with distance, for f and g in Y^X,

$$Y^X(f,g) = \sup\{Y(f(x), g(x)) \mid x \in X\}.$$

The product $X \times Y$ of two gum's X and Y is defined as the Cartesian product of their underlying sets, together with distance, for $\langle x,y \rangle$ and $\langle x',y' \rangle$ in $X \times Y$,

$$X \times Y(\langle x,y \rangle,\ \langle x',y' \rangle) = \max\{X(x,x'),\ Y(y,y')\}.$$

From the above definition it follows that for every gum X, its distance function $X(-,-)$ is a non-expansive mapping from the gum $X^{op} \times X$ to the gum $[0,1]$, where X^{op} is the gum *opposite* to X defined as the set X together with the distance $X^{op}(x,x') = X(x',x)$.

A *preorder* is a set P together with a binary relation $\leq$ on P which is reflexive and transitive. A preorder P can be viewed as a gum, by defining

$$P(p,q) = \begin{cases} 0 & \text{if } p \leq q \\ 1 & \text{if } p \not\leq q. \end{cases}$$

Note that if P is a partial order then this defines a quasi ultrametric and that the non-expansive mappings between preorders are precisely the monotone maps. By a slight abuse of language, any gum stemming from a preorder in this way will itself be called a preorder. Conversely, any gum X gives rise to a preorder $\langle X, \leq_X \rangle$, where $\leq_X$, called the *underlying* ordering of X, is given, for x and y in X, by

$$x \leq_X y \text{ if and only if } X(x,y) = 0.$$

For instance, the ordering that underlies $[0,1]$ is the reverse of the usual ordering: for r and s in $[0,1]$,

$$r \leq_{[0,1]} s \text{ if and only if } s \leq r.$$

Any (pseudo or quasi) ultrametric space is a fortiori a gum. Conversely, any gum X induces a pseudo ultrametric space by taking the symmetrization of X ([Rut95]).

3 A generalized Alexandroff topology

For a gum X, let $\hat{X}$ denote the function space

$$\hat{X} = [0,1]^{X^{op}}.$$

An element ϕ in $\hat{X}$ can be interpreted as a 'fuzzy' predicate (or 'fuzzy' subset) on X (cf. [Law86]): the value that ϕ assigns to an element x in X is thought of as a measure for 'the extent to which x is an element of ϕ'. The smaller this number is, the more x should be viewed as an element of ϕ. Every gum can be isometrically embedded into the set of its 'fuzzy' predicates.

Lemma 3.1 (Yoneda Lemma) *Let X be a gum and x in X. The function*

$$X(-,x) : X^{op} \to [0,1]$$

mapping y in X^{op} to $X(y,x)$ is non-expansive and hence an element of $\hat{X}$. Furthermore,

$$\hat{X}(X(-,x),\phi) = \phi(x)$$

for any function ϕ in $\hat{X}$.

Proof: For x in X, $X(-,x)$ is non-expansive since, for y and z in X,

$$[0,1](X(y,x),\ X(z,x)) \leq X(z,y) = X^{op}(y,z)$$

follows from $X(z,x) \leq \max\{X(z,y), X(y,x)\}$ by Proposition 2.1. Now let ϕ in $\hat{X}$. On the one hand,

$$\begin{aligned} \phi(x) &= [0,1](X(x,x),\phi(x)) \\ &\leq \sup\{[0,1](X(y,x),\phi(y)) \mid y \in X\} \\ &= \hat{X}(X(-,x),\phi). \end{aligned}$$

On the other hand, non-expansiveness of ϕ gives, for any y in X,

$$[0,1](\phi(x),\phi(y)) \leq X^{op}(x,y) = X(y,x)$$

which is equivalent to $[0,1](X(y,x),\phi(y)) \leq \phi(x)$ by Proposition 2.1. $\square$

The following corollary is immediate.

Corollary 3.2 *For a gum X, the Yoneda embedding $\mathbf{y} : X \to \hat{X}$, defined, for x in X, by*

$$\mathbf{y}(x) = X(-,x)$$

is isometric, that is, $X(x,x') = \hat{X}(\mathbf{y}(x),\mathbf{y}(x'))$ for all x and x' in X. $\square$

Given a fuzzy predicate ϕ in $\hat{X}$, by taking only its 'real' elements, i.e. the elements x of X for which $\phi(x) = 0$, we obtain its *extension*

$$\textstyle\int_A \phi = \{x \in X \mid \phi(x) = 0\},$$

where the subscript A stands for Alexandroff. Notice that for any ϕ in $\hat{X}$,

$$\begin{aligned} \textstyle\int_A \phi &= \{x \in X \mid \phi(x) = 0\} \\ &= \{x \in X \mid \hat{X}(\mathbf{y}(x),\phi) = 0\} \quad \text{[the Yoneda Lemma 3.1]} \\ &= \{x \in X \mid \mathbf{y}(x) \leq_{\hat{X}} \phi\}. \end{aligned}$$

Hence by Corollary 3.2, for any y in X,

$$\textstyle\int_A \mathbf{y}(y) = \{x \in X \mid \mathbf{y}(x) \leq_{\hat{X}} \mathbf{y}(y)\} = \{x \in X \mid x \leq_X y\} = y\downarrow.$$

Any subset $V \subseteq X$ defines a predicate $\rho_A(V) : X^{op} \to [0,1]$ which is referred to as the *character* of the subset V. It is defined, for x in X, by

$$\rho_A(V)(x) = \inf\{X(x,v) \mid v \in V\},$$

i.e., the distance from x to the set V. Notice that $\rho_A(V)(x) = \inf\{\mathbf{y}(v)(x) \mid v \in V\}$ for any x in X.

These two constructions define mappings $\int_A : \hat{X} \to \mathcal{P}(X)$ and $\rho_A : \mathcal{P}(X) \to \hat{X}$, which can be nicely related by considering $\hat{X}$ with the underlying preorder $\leq_{\hat{X}}$, and $\mathcal{P}(X)$ ordered by subset inclusion (cf. [Law86]):

Proposition 3.3 *Assume X is a gum. Then the mappings $\int_A : \langle \hat{X}, \leq_{\hat{X}} \rangle \to \langle \mathcal{P}(X), \subseteq \rangle$ and $\rho_A : \langle \mathcal{P}(X), \subseteq \rangle \to \langle \hat{X}, \leq_{\hat{X}} \rangle$ are monotone. Moreover ρ_A is left adjoint to $\int_A$.*

Proof: Monotonicity of $\int_A$ and ρ_A follows directly from their definitions. We will hence concentrate on the second part of the proposition by proving for all V in $\mathcal{P}(X)$ and ϕ in $\hat{X}$,

$$V \subseteq \textstyle\int_A \rho_A(V) \quad \text{and} \quad \rho_A(\textstyle\int_A \phi) \leq_{\hat{X}} \phi,$$

which is equivalent to ρ_A being left adjoint to $\int_A$, (cf. Theorem 0.3.6 of [GHK+80]). For V in $\mathcal{P}(X)$ we have

$$\textstyle\int_A \rho_A(V) = \{x \in X \mid \mathbf{y}(x) \leq_{\hat{X}} \rho_A(V)\} \supseteq V,$$

because for all $v \in V$, $\mathbf{y}(v) \leq_{\hat{X}} \rho_A(V)$. Furthermore, for ϕ in $\hat{X}$ and x in X,

$$\begin{array}{rcl} \rho_A(\int_A \phi)(x) & = & \inf\{X(x,v) \mid v \in X \text{ and } \mathbf{y}(v) \leq_{\hat{X}} \phi\} \\ & = & \inf\{\mathbf{y}(v)(x) \mid v \in X \text{ and } \forall z \in X, \mathbf{y}(v)(z) \geq \phi(z)\} \\ & \geq & \inf\{\mathbf{y}(v)(x) \mid v \in X \text{ and } \mathbf{y}(v)(x) \geq \phi(x)\} \\ & \geq & \phi(x). \end{array}$$

Consequently, $\rho_A(\int_A \phi) \leq_{\hat{X}} \phi$. □

The above fundamental adjunction relates character of subsets and extension of predicates and is often referred to as the *comprehension schema* [Law73, Ken87]. As with any adjoint pair between preorders (cf. Theorem 0.3.6 of [GHK+80]), the composition $\int_A \circ \rho_A$ is a closure operator on X. It satisfies, for $V \subseteq X$,

$$\begin{array}{rcll} \int_A \circ \rho_A(V) & = & \{x \in X \mid \rho_A(V)(x) = 0\} & \\ & = & \{x \in X \mid \hat{X}(\mathbf{y}(x), \rho_A(V)) = 0\} & \text{[the Yoneda Lemma 3.1]} \\ & = & \{x \in X \mid \forall z \in X,\ \ [0,1](\mathbf{y}(x)(z), \rho_A(V)(z)) = 0\} & \\ & = & \{x \in X \mid \forall z \in X,\ \ \mathbf{y}(x)(z) \geq \rho_A(V)(z)\} & \\ & = & \{x \in X \mid \forall \epsilon > 0\, \forall z \in X,\ \ \mathbf{y}(x)(z) < \epsilon \Rightarrow (\exists v \in V,\ X(z,v) < \epsilon)\} & \\ & = & \{x \in X \mid \forall \epsilon > 0\, \forall z \in X,\ \ X(z,x) < \epsilon \Rightarrow (\exists v \in V,\ X(z,v) < \epsilon)\} & (1) \end{array}$$

By using the above characterization we can prove the following lemma.

Lemma 3.4 *For a gum X, the closure operator $\int_A \circ \rho_A : \mathcal{P}(X) \to \mathcal{P}(X)$ is topological.*

Proof: It is an immediate consequence of (1) that $\int_A \circ \rho_A(\emptyset) = \emptyset$. Moreover, for $V, W \subseteq X$,

$$\textstyle\int_A \circ \rho_A(V \cup W) \quad \supseteq \quad \int_A \circ \rho_A(V) \cup \int_A \circ \rho_A(W),$$

because $\int_A \circ \rho_A$ is a closure operator. For the reverse inclusion, let x in $\int_A \circ \rho_A(V \cup W)$. Suppose $x \notin \int_A \circ \rho_A(V)$. We will show x in $\int_A \circ \rho_A(W)$: consider $\epsilon_1 > 0$ and z_1 in X with

$X(z_1, x) < \epsilon_1$. We should find y in W with $X(z_1, y) < \epsilon_1$. Because $x \notin \int_A \circ \rho_A(V)$ there exist $\epsilon_0 > 0$ and z_0 in X such that

$$X(z_0, x) < \epsilon_0 \quad \text{and} \quad (\forall y \in V,\ X(z_0, y) \geq \epsilon_0). \tag{2}$$

Let $\epsilon = \min\{\epsilon_0, \epsilon_1\}$. Because x in $\int_A \circ \rho_A(V \cup W)$ and $X(x, x) = 0 < \epsilon$, there exists y in $V \cup W$ with $X(x, y) < \epsilon$. The assumption that y in V contradicts (2), because

$$X(z_0, y) \leq \max\{X(z_0, x), X(x, y)\} < \max\{\epsilon_0, \epsilon\} = \epsilon_0.$$

Thus y in W. Furthermore,

$$X(z_1, y) \leq \max\{X(z_1, x), X(x, y)\} < \max\{\epsilon_1, \epsilon\} = \epsilon_1.$$

□

The above lemma implies that the closure operator $\int_A \circ \rho_A$ induces a topology on X. In Proposition 3.6 below, it is proved equivalent to the following generalized ϵ-ball topology: For x in X and $\epsilon > 0$ define the *ϵ-ball centered in x* by

$$B_\epsilon(x) = \{z \in X \mid X(x, z) < \epsilon\}.$$

A subset V of a gum X is *generalized Alexandroff* open (gA-open, for short) if, for x in X,

$$x \in V \quad \Rightarrow \exists \epsilon > 0,\ B_\epsilon(x) \subseteq V.$$

The set of all gA-open subsets of X is denoted by $\mathcal{O}_{gA}(X)$. For instance, for every x in X, the ϵ-ball $B_\epsilon(x)$ is a gA-open set. The pair $\langle X, \mathcal{O}_{gA}(X)\rangle$ can be shown to be a topological space with $\{B_\epsilon(x) \mid \epsilon > 0 \text{ and } x \in X\}$ as basis (cf. [FK95]).

Every topology $\mathcal{O}(X)$ on a set X induces a preorder on X called the *specialization* preorder: for any x and y in X, $x \leq_{\mathcal{O}} y$ if and only if $\forall V \in \mathcal{O}(X),\ \ x \in V \Rightarrow y \in V$. The specialization preorder on a gum X induced by its generalized Alexandroff topology coincides with the preorder underlying X:

Proposition 3.5 *Let X be a gum. For x and y in X, $x \leq_{\mathcal{O}_{gA}} y$ if and only if $x \leq_X y$.*

Proof: For any gA-open set V, if $x \in V$ and $X(x, y) = 0$ then $y \in V$. From this observation the implication from right to left is clear. For the converse suppose $x \leq_{\mathcal{O}_{gA}} y$. Then for every $\epsilon > 0$, $x \in B_\epsilon(x)$ implies $y \in B_\epsilon(x)$ because ϵ-balls are gA-open sets. Hence $X(x, y) < \epsilon$. Since ϵ was arbitrary, $X(x, y) = 0$, that is $x \leq_X y$. □

In the following proposition we use the fact that for every topology $\mathcal{O}(X)$, the induced topological closure operator cl on X can be characterized as follows: For every $V \subseteq X$, $cl(V) = V \cup V^d$, where V^d is the *derived set*, that is, the set of all accumulation points of V. For a subset V of X we write $cl_A(V)$ for the closure of V in the generalized Alexandroff topology.

Proposition 3.6 *For every subset V of a gum X, $cl_A(V) = \int_A \circ \rho_A(V)$.*

Proof: It follows from the characterization (1) of $\int_A \circ \rho_A$ and because $cl_A(V) = V \cup V^d$ that it is sufficient to prove

$$V \cup V^d \;=\; \{x \in X \mid \forall \epsilon > 0\; \forall z \in X,\;\; X(z,x) < \epsilon \;\Rightarrow\; (\exists v \in V,\; X(z,v) < \epsilon)\}. \qquad (3)$$

From the definition of accumulation point and the fact that the set of all ϵ-balls is a basis for the generalized Alexandroff topology, it follows that for every x in X,

$$\begin{array}{rcl} x \in V^d & \iff & \forall W \in \mathcal{O}_{gA}(X),\;\; x \in o \;\Rightarrow\; W \cap (V \setminus \{x\}) \neq \emptyset \\ & \iff & \forall \epsilon > 0\; \forall z \in X,\;\; x \in B_\epsilon(z) \;\Rightarrow\; B_\epsilon(z) \cap (V \setminus \{x\}) \neq \emptyset \\ & \iff & \forall \epsilon > 0\; \forall z \in X,\;\; X(z,x) < \epsilon \;\Rightarrow\; \exists v \in (V \setminus \{x\}),\; X(z,v) < \epsilon. \end{array}$$

Therefore (3) holds. □

For ordinary ultrametric spaces, gA-open sets are just the usual open sets. For preorders, a set is gA-open precisely when it is Alexandroff open (upper closed) because if X is a preorder then for $\epsilon \leq 1$,

$$\begin{array}{rcl} B_\epsilon(x) & = & \{y \in X \mid X(x,y) < \epsilon\} \\ & = & \{y \in X \mid X(x,y) = 0\} \\ & = & \{y \in X \mid x \leq_X y\} \\ & = & x\!\uparrow, \end{array}$$

while in case $\epsilon > 1$ then $B_\epsilon(x) = X$, which is clearly upper closed.

4 Cauchy sequences, limits, and completeness

For computational reasons we are interested in complete spaces, in which one can model infinite behaviors. For this aim, Cauchy sequences for gum's are introduced. It is explained how such sequences look like in $[0,1]$, and how to define in $[0,1]$ the notion of limit. This will give rise to a definition of limit for arbitrary gum's. Furthermore the notions of completeness, finiteness, and algebraicity are introduced.

A sequence $(x_n)_n$ in a gum X is *forward-Cauchy* if

$$\forall \epsilon > 0\; \exists N\; \forall n \geq N,\;\; X(x_n, x_{n+1}) \leq \epsilon.$$

Note that this is equivalent to the more familiar condition:

$$\forall \epsilon > 0\; \exists N\; \forall n \geq m \geq N,\;\; X(x_m, x_n) \leq \epsilon,$$

because of the strong triangle inequality. Since our metrics need not be symmetric, the following variation exists: a sequence $(x_n)_n$ is *backward-Cauchy* if

$$\forall \epsilon > 0\; \exists N\; \forall n \geq N,\;\; X(x_{n+1}, x_n) \leq \epsilon.$$

If X is an ordinary ultrametric space then forward-Cauchy and backward-Cauchy both mean Cauchy in the usual sense. And if X is a preorder then forward-Cauchy sequences are eventually increasing: there exists an N such that for all $n \geq N$, $x_n \leq_X x_{n+1}$. Increasing sequences in a preorder are usually called ω-chains. Similarly backward-Cauchy sequences are eventually decreasing.

Forward-Cauchy sequences in $[0,1]$, with the generalized ultrametric of Section 2, are particularly simple: every forward-Cauchy sequence either converges to 0 or is eventually decreasing; dually, every backward-Cauchy sequence either converges to 0 or is eventually increasing.

Proposition 4.1 *A sequence $(r_n)_n$ in $[0,1]$ is forward-Cauchy if and only if*

$$\forall \epsilon > 0\ \exists N\ \forall n \geq N, \quad r_n \leq \epsilon \ \textit{or} \ \exists N\ \forall n \geq N, \quad r_n \geq r_{n+1}.$$

Dually, it is backward-Cauchy if and only if

$$\forall \epsilon > 0\ \exists N\ \forall n \geq N, \quad r_n \leq \epsilon \ \textit{or} \ \exists N\ \forall n \geq N, \quad r_n \leq r_{n+1}.$$

Proof: We prove only the 'only if' implication of the first statement. Let $(r_n)_n$ be forward-Cauchy in $[0,1]$. Suppose there exists $\epsilon > 0$ such that

$$\forall N\ \exists n \geq N, \quad r_n > \epsilon.$$

We claim that there exists an N such that for all $n \geq N$, $r_n > \epsilon$; for suppose not:

$$\forall N\ \exists n \geq N, \quad r_n \leq \epsilon.$$

Because $(r_n)_n$ is forward-Cauchy, there exists M such that for all $m \geq M$, $[0,1](r_m, r_{m+1}) \leq \epsilon$. Consider $n_1 \geq M$ with $r_{n_1} \leq \epsilon$, and consider $n_2 \geq n_1$ with $r_{n_2} > \epsilon$. Then

$$\begin{array}{lll} \epsilon & < & r_{n_2} \\ & = & [0,1](r_{n_1}, r_{n_2}) \quad \text{[definition distance on } [0,1]] \\ & \leq & \epsilon, \end{array}$$

a contradiction. Therefore let N be such that for all $n \geq N$, $r_n > \epsilon$. Let $M \geq N$ such that for all $m \geq M$, $[0,1](r_m, r_{m+1}) \leq \epsilon$, which is equivalent to $r_{m+1} \leq \max\{\epsilon, r_m\}$ (by Proposition 2.1). Because $r_m > \epsilon$, for all $m \geq M$, this implies $r_{m+1} \leq r_m$. $\square$

Because forward- and backward-Cauchy sequences in $[0,1]$ are that simple, the following definitions are easy as well: the *forward-limit* of a forward-Cauchy sequence $(r_n)_n$ in $[0,1]$ is given by

$$\underset{\rightarrow}{\lim}\, r_n = \sup_n \inf_{k \geq n} r_k.$$

Dually, the *backward-limit* of a backward-Cauchy sequence $(r_n)_n$ in $[0,1]$ is

$$\underset{\leftarrow}{\lim}\, r_n = \inf_n \sup_{k \geq n} r_k.$$

These numbers are what one intuitively would consider as limits of Cauchy sequences. Forward-limits and backward-limits in $[0,1]$ are related as follows: Let $(r_n)_n$ be a forward-Cauchy sequence in $[0,1]$. Then for all r in $[0,1]$,

$$[0,1](\underrightarrow{\lim}\, r_n, r) \;=\; \underleftarrow{\lim}[0,1](r_n, r). \tag{4}$$

Forward-limits in an *arbitrary* gum X can now be defined in terms of backward-limits in $[0,1]$: an element x is a *forward-limit* of a forward-Cauchy sequence $(x_n)_n$ in X,

$$x = \underrightarrow{\lim}\, x_n \text{ if and only if } \forall y \in X,\;\; X(x,y) = \underleftarrow{\lim}\, X(x_n, y).$$

This is well defined because if $(x_n)_n$ is forward-Cauchy in X, then $(X(x_n,y))_n$ is backward-Cauchy in $[0,1]$, for any y in X. Note that our earlier definition of the forward-limit of forward-Cauchy sequences in $[0,1]$ is consistent with this definition for arbitrary gum's: this follows from (4).

For ordinary ultrametric spaces, the above defines the usual notion of limit:

$$x = \underrightarrow{\lim}\, x_n \text{ if and only if } \forall \epsilon > 0\; \exists N\; \forall n \geq N,\;\; X(x_n, x) < \epsilon.$$

If X is a preorder and $(x_n)_n$ is an ω-chain in X then

$$x = \underrightarrow{\lim}\, x_n \text{ if and only if } \forall y \in X,\;\; x \leq_X y \Leftrightarrow \forall n,\; x_n \leq_X y,$$

i.e., $x = \bigsqcup x_n$, the least upper-bound of the ω-chain $(x_n)_n$.

One could also consider backward-limits for arbitrary gum's. Since these will not play a role in the rest of this paper, these are omitted. For simplicity, we shall use *Cauchy* instead of forward-Cauchy. Similarly, we shall write $\lim x_n$ for $\underrightarrow{\lim}\, x_n$.

Cauchy sequences may have more than one limit. All limits have distance 0, however. As a consequence, limits are unique in quasi ultrametric spaces.

A gum X is *complete* if every Cauchy sequence in X has a limit. For instance, $[0,1]$ is complete. It follows that a preorder is complete if and only if all its ω-chains have a least upper-bound. Complete partial orders are called cpo. For ordinary ultrametric spaces this definition of completeness is the usual one. There is the following fact (cf. Theorem 6.5 of [Rut95]).

Proposition 4.2 *Let X and Y be gum's. If Y is complete then Y^X is complete. Moreover, limits are pointwise: let $(f_n)_n$ be a Cauchy sequence in Y^X and f an element in Y^X. Then $\lim f_n = f$ if and only if for all x in X, $\lim f_n(x) = f(x)$.* $\square$

The above proposition implies that for every gum X the function space $\hat{X}$ is complete. Therefore by using the Yoneda embedding of Corollary 3.2, we have that every gum can be isometrically embedded in a complete gum. This fact is used in [BBR95] to define the completion of a gum X.

A mapping $f : X \to Y$ between gum's X and Y is *continuous* if it preserves limits: if $x = \lim x_n$ in X then $f(x) = \lim f(x_n)$ in Y. For ordinary ultrametric spaces, this is the usual definition. For preorders it means preservation of least upper-bounds.

For every gum X, the distance function $X(-,-) : X^{op} \times X \to [0,1]$ is (backward) continuous in its first argument by the definition of limit. In general, however, it is not continuous in its second argument. An element a in a gum X is *finite* if the function $X(a,-) : X \to [0,1]$, mapping every x in X to $X(a,x)$, is continuous. If X is a preorder this means that for any ω-chain $(x_n)_n$ in X,

$$X(a, \bigsqcup x_n) = \lim X(a, x_n),$$

or, equivalently, $a \leq_X \bigsqcup x_n$ if and only if $\exists n, \quad a \leq_X x_n$, which is the usual definition of finiteness for ordered spaces. If X is an ordinary ultrametric space then $X(a,-)$ is continuous for any a in X, hence all elements are finite. The following lemma gives an example of finite elements in the function space $\hat{X}$:

Lemma 4.3 *For any gum X and x in X, $\mathbf{y}(x)$ is finite in $\hat{X}$.*

Proof: We have to show that $\hat{X}(\mathbf{y}(x),-) : \hat{X} \to [0,1]$ is continuous: for any Cauchy sequence $(\phi_n)_n$ in $\hat{X}$,

$$\begin{aligned}
\hat{X}(\mathbf{y}(x), \lim \phi_n) &= (\lim \phi_n)(x) \quad \text{[the Yoneda Lemma 3.1]} \\
&= \lim \phi_n(x) \quad \text{[Proposition 4.2]} \\
&= \lim \hat{X}(\mathbf{y}(x), \phi_n) \quad \text{[the Yoneda Lemma 3.1].}
\end{aligned}$$

□

A *basis* for X is a subset $B \subseteq X$ consisting of finite elements such that every element x in X is the limit $x = \lim a_n$ of a Cauchy sequence $(a_n)_n$ of elements in B. A gum X is *algebraic* if there exists a collection B of finite elements of X which is a basis for X.

5 A generalized Scott topology

A topology for a complete space X can then be considered satisfactory if limits in X are topological limits. This is not the case for the generalized Alexandroff topology: for instance, for complete partial orders $\mathcal{O}_{gA}(X)$ coincides with the standard Alexandroff topology, for which the coincidence of the least upper-bounds of ω-chains and their topological limits does not hold. Therefore the *Scott* topology is usually considered to be preferable: it is the coarsest topology refining the Alexandroff topology, in which least upper bounds of ω-chains are topological limits (cf. [GHK+80] and [Smy92]). Also for gum's, a suitable refinement of the generalized Alexandroff topology exists. A key step towards its definition is the following restriction of the Yoneda embedding to a continuous and isometric function.

Theorem 5.1 *Let X be a complete gum. If $B \subseteq X$ is a basis for X then the function $\mathbf{y}_B : X \to \hat{B}$ defined, for x in X, by*

$$\mathbf{y}_B(x) = \lambda b \in B \,.\, X(b,x)$$

is isometric and continuous.

Proof: According to Corollary 3.2, $\mathbf{y}$ is isometric. Consequently, $\mathbf{y}_B$ is non-expansive. Because, for all Cauchy sequences $(x_n)_n$ in X,

$$\begin{aligned}
\lim_n \mathbf{y}_B(x_n) &= \lim_n \lambda b \in B \,.\, X(b, x_n) \\
&= \lambda b \in B \,.\, \lim_n X(b, x_n) \quad \text{[Proposition 4.2]} \\
&= \lambda b \in B \,.\, X(b, \lim_n x_n) \quad [b \text{ is finite in } X] \\
&= \mathbf{y}_B\,(\lim_n x_n),
\end{aligned}$$

$\mathbf{y}_B$ is continuous. It is also isometric because, for x and y in X, since B is a basis there exist Cauchy sequences $(a_n)_n$ and $(b_m)_m$ in B such that $x = \lim_n a_n$ and $y = \lim_m b_m$. Hence

$$\begin{aligned}
\hat{B}(\mathbf{y}_B(x), \mathbf{y}_B(y)) &= \hat{B}(\mathbf{y}_B(\lim_n a_n), \mathbf{y}_B(\lim_m b_m)) \\
&= \hat{B}(\lim_n \mathbf{y}_B(a_n), \lim_m \mathbf{y}_B(b_m)) \quad [\mathbf{y}_B \text{ is continuous}] \\
&= \lim_{\leftarrow n} \hat{B}(\mathbf{y}_B(a_n), \lim_m \mathbf{y}_B(b_m)) \quad \text{[definition of limit]} \\
&= \lim_{\leftarrow n} \lim_m \hat{B}(\mathbf{y}_B(a_n), \mathbf{y}_B(b_m)) \quad [\mathbf{y}_B(a_n) \text{ is finite in } \hat{B} \text{ (Lemma 4.3)}] \\
&= \lim_{\leftarrow n} \lim_m B(a_n, b_m) \quad [\mathbf{y}_B \text{ is isometric (Corollary 3.2)}] \\
&= \lim_{\leftarrow n} \lim_m X(a_n, b_m) \quad [B \subseteq X] \\
&= \lim_{\leftarrow n} X(a_n, y) \quad [a_n \text{ is finite in } X] \\
&= X(x, y) \quad \text{[definition of limit]}
\end{aligned}$$

□

The converse of the above theorem holds as well [BBR95, Theorem 5.6]. For an algebraic complete gum X with basis B we can now consider the collection $\hat{B}$ of fuzzy subsets of B rather than the collection $\hat{X}$ as we have done in Section 3. These fuzzy subsets can be compared with subsets of X by the extension function $⨏_S : \hat{B} \to \mathcal{P}(X)$ and the character function $\rho_S : \mathcal{P}(X) \to \hat{B}$ defined, for ϕ in $\hat{B}$ and $V \subseteq X$, by

$$⨏_S \phi = \{x \in X \mid \mathbf{y}_B(x) \leq_{\hat{B}} \phi\} \quad \text{and} \quad \rho_S(V) = \lambda b \in B. \inf\{X(b, v) \mid v \in V\}.$$

As in Proposition 3.3 the maps $⨏_S : \langle \hat{B}, \leq_{\hat{B}} \rangle \to \langle \mathcal{P}(X), \subseteq \rangle$ and $\rho_S : \langle \mathcal{P}(X), \subseteq \rangle \to \langle \hat{B}, \leq_{\hat{B}} \rangle$ are monotone and ρ_S is left adjoint to $⨏_S$. Thus, $⨏_S \circ \rho_S : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator which can, in a way similar to (1), be characterized, for an algebraic complete gum X with basis B, as follows: for $V \subseteq X$,

$$⨏_S \circ \rho_S(V) \;=\; \{x \in X \mid \forall \epsilon > 0 \; \forall a \in B, \;\; X(a, x) < \epsilon \Rightarrow (\exists y \in V, \; X(a, y) < \epsilon)\}. \quad (5)$$

Also this closure operator is topological:

Lemma 5.2 *For an algebraic complete gum X, the closure operator $⨏_S \circ \rho_S : \mathcal{P}(X) \to \mathcal{P}(X)$ is topological.*

Proof: This lemma is proved along the same lines as Lemma 3.4, but one needs the following additional observation: If $B \subseteq X$ is a basis for X then for any z_0 and z_1 in B, ϵ_0, $\epsilon_1 > 0$, and x in X, such that

$$X(z_0, x) < \epsilon_0 \text{ and } X(z_1, x) < \epsilon_1,$$

there exists b in B such that $X(z_0, b) < \epsilon_0$, $X(z_1, b) < \epsilon_1$, and $X(b, x) < \min\{\epsilon_0, \epsilon_1\}$. This fact can be proved as follows. Because X is an algebraic complete gum with B as basis, there exists a Cauchy sequence $(b_n)_n$ in B with $x = \lim b_n$. Since z_0 in B, it is finite in X. Hence, $X(z_0, x) = X(z_0, \lim b_n) < \epsilon_0$ implies the existence of N_0 such that for all $n \geq N_0$, $X(z_0, b_n) < \epsilon_0$. Similarly, there exists N_1 such that for all $n \geq N_1$, $X(z_1, b_n) < \epsilon_1$. Furthermore, there exists, by definition of limit, N_2 such that for all $n \geq N_2$, $X(b_n, x) < \min\{\epsilon_0, \epsilon_1\}$. By taking $M = \max\{N_0, N_1, N_2\}$, and putting $b = b_M$, we have found the element in X we were looking for. □

Thus the closure operator above induces a topology on X which we will call the *generalized Scott* topology. Indeed, in the special case that X is an algebraic complete partial order with basis B, for every $V \subseteq X$,

$$\textstyle\int_S \circ \rho_S(V) \quad = \quad \{x \in X \mid \forall a \in B, \ \ a \leq_X x \Rightarrow (\exists v \in V, \ a \leq_X v)\},$$

which we recognize as the closure operator induced by the Scott topology on X.

Next an alternative definition of the generalized Scott topology is given by specifying the open sets (this time starting with a complete gum X). In Theorem 5.6 below, it will be shown that the closure operator induced by this second definition coincides with $\int_S \circ \rho_S$ whenever X is algebraic.

A subset V of a complete gum X is *generalized Scott* open (gS-open, for short) if for all Cauchy sequences $(x_n)_n$ in X,

$$\lim x_n \in V \Rightarrow \exists N \; \exists \epsilon > 0 \; \forall n \geq N, \ \ B_\epsilon(x_n) \subseteq V.$$

The set of all gS-open subsets of X is denoted by $\mathcal{O}_{gS}(X)$. Below it will be shown that this defines a topology indeed. Note that every gS-open set $V \subseteq X$ is gA-open because every point x of a gum X is the limit of the constant Cauchy sequence $(x)_n$. Therefore this topology refines the generalized Alexandroff topology. Furthermore it will be shown to coincide with the ϵ-ball topology in case X is a complete ultrametric space; and to coincide with the Scott topology in case X is a cpo. The following proposition gives an example of gS-open sets.

Proposition 5.3 *For every complete gum X, an element a in X is finite if and only if for every $\epsilon > 0$, the set $B_\epsilon(a)$ is gS-open.*

Proof: Let a in X be finite and $\epsilon > 0$. Then the ϵ-ball $B_\epsilon(a)$ is a gS-open set: let $(x_n)_n$ be a Cauchy sequence in X and assume $\lim x_n$ in $B_\epsilon(a)$. Because a is finite, $X(a, \lim x_n) = \lim X(a, x_n) < \epsilon$, by which there exists $\gamma > 0$ such that $\lim X(a, x_n) < \epsilon - \gamma$. Take $\delta < \gamma$.

Then there exists N such that $X(a,x_n) < (\epsilon - \gamma) + \delta$, for all $n \geq N$. Then $B_\delta(x_n) \subseteq B_\epsilon(a)$ for all $n \geq N$, because if y in $B_\delta(x_n)$ for some $n \geq N$ we have, by the strong triangular inequality and by our choice of δ,

$$X(a,y) \leq \max\{X(a,x_n), X(x_n,y)\} < \max\{((\epsilon-\gamma)+\delta), \delta\} = (\epsilon-\gamma)+\delta < \epsilon,$$

that is, y in $B_\epsilon(a)$. Conversely, assume $B_\epsilon(a)$ is a gS-open set for every $\epsilon > 0$. We only need to prove, for every Cauchy sequence $(x_n)_n$ in X, that

$$\lim X(a,x_n) \leq X(a, \lim x_n) \tag{6}$$

because the reverse inequality holds whether a is finite or not (being the the distance function $X(-,-) : X^{op} \times X \to [0,1]$ non-expansive). If $\lim X(a,x_n) = 0$ then (6) is trivially true. Therefore suppose $\lim X(a,x_n) > 0$ and, towards a contradiction, assume $X(a,\lim x_n) < \lim X(a,x_n)$. Then there exists $\epsilon > 0$ such that $X(a,\lim x_n) < \epsilon < \lim X(a,x_n)$. Moreover

$$\begin{array}{rcll} X(a,\lim x_n) < \epsilon & \Rightarrow & \lim x_n \in B_\epsilon(a) & \\ & \Rightarrow & \exists N \; \exists \delta > 0 \; \forall n \geq N, \quad B_\delta(x_n) \subseteq B_\epsilon(a) & [B_\epsilon(a) \text{ is gS-open}] \\ & \Rightarrow & \exists N \; \forall n \geq N, X(a,x_n) < \epsilon & \\ & \iff & \lim X(a,x_n) < \epsilon. & \end{array}$$

But this contradicts $\epsilon < \lim X(a,x_n)$. Thus $X(a,\lim x_n) \geq \lim X(a,x_n)$. □

The collection of all gS-open sets forms indeed a topology:

Proposition 5.4 *For every complete gum X, $\mathcal{O}_{gS}(X)$ is a topology on X. If X is also algebraic with basis B, then the set $\{B_\epsilon(a) \mid a \in B \text{ and } \epsilon > 0\}$ forms a basis for the generalized Scott topology $\mathcal{O}_{gS}(X)$.*

Proof: One can easily verify that $\mathcal{O}_{gS}(X)$ is closed under finite intersections and arbitrary unions. We will only prove that, for an algebraic complete gum X with base B, every gS-open set $V \subseteq X$ is the union of ϵ-balls of finite elements. Let x in V. Since X is algebraic there is a Cauchy sequence $(a_n)_n$ in B with $x = \lim a_n$. Because V is gS-open, there exists $\epsilon_x > 0$ and $N_x \geq 0$ such that $B_{\epsilon_x}(a_n) \subseteq V$ for all $n \geq N_x$ and with x in $B_{\epsilon_x}(a_n)$ for N_x big enough. Therefore $V \subseteq \bigcup_{x \in V} B_{\epsilon_x}(a_{N_x})$. Since the other inclusion trivially holds we have that the collection of all ϵ-balls of finite elements forms a basis for the generalized Scott topology. □

Any ordinary complete ultrametric space X is an algebraic complete gum in which all elements are finite. Therefore, by the previous proposition, the basic open sets of the generalized Scott topology are all the ϵ-balls $B_\epsilon(x)$, with x in X. Hence for ordinary complete ultrametric spaces the generalized Scott topology coincides with the standard ϵ-ball topology.

For a cpo X, a set $V \subseteq X$ is gS-open precisely when it is Scott open, that is, V is upper closed and moreover satisfies, for any ω-chain $(x_n)_n$ in X,

$$\bigsqcup x_n \in V \;\Rightarrow\; \exists N,\;\; x_N \in V.$$

Indeed, if $V \subseteq X$ is gS-open then it is upper closed because the generalized Scott topology refines the generalized Alexandroff topology. Moreover, if $\bigsqcup x_n$ in V for an ω-chain $(x_n)_n$ in X then there exists $\epsilon > 0$ and N such that $B_\epsilon(x_n) \subseteq V$ for all $n \geq N$. Since x_n in $B_\epsilon(x_n)$ for all ϵ, V is an ordinary Scott open set. Conversely, assume V is Scott open and let $\lim x_n$ in V. Because V is Scott open and limits are least upper bounds there exists N such that x_n in V for all $n \geq N$. By taking $\epsilon = 1/2$ we obtain that V is also gS-open because for every x in X, $B_{1/2}(x) = x\!\uparrow$.

A subset V of a complete gum X is *generalized Scott closed* (gS-closed, for short) if its complement $X \setminus V$ is gS-open. This is equivalent to the following condition: for all Cauchy sequences $(x_n)_n$ in X,

$$(\forall N\; \forall \epsilon > 0\; \exists n \geq N\; \exists y \in V,\;\; X(x_n, y) < \epsilon) \;\Rightarrow\; \lim x_n \in V.$$

Notice that if V is a gS-closed set and $(x_n)_n$ is a Cauchy sequence in V (satisfying hence the above condition), then all its limits should belong to V. The following lemma, due to Flagg and Sünderauf, gives an example of gS-closed sets.

Lemma 5.5 *For every complete gum X, x in X and $\delta \geq 0$ the set $\bar{B}^{op}_\delta(x) = \{y \in X \mid X(y, x) \leq \delta\}$ is gS-closed.*

Proof: Let $(x_n)_n$ be a Cauchy sequence in X such that

$$\forall N\; \forall \epsilon > 0\; \exists n \geq N\; \exists y \in \bar{B}^{op}_\delta(x),\;\; X(x_n, y) < \epsilon.$$

Then

$$\forall N\; \forall \epsilon > 0\; \exists n \geq N,\;\; X(x_n, y) \leq \max\{\epsilon, \delta\}. \tag{7}$$

We need to show that $\lim x_n$ in $\bar{B}^{op}_\delta(x)$; that is, $X(\lim x_n, x) \leq \delta$. Suppose $\epsilon > 0$. Since $(x_n)_n$ is a Cauchy sequence we can choose N_0 so that $\forall m \geq n \geq N_0$, $X(x_n, x_m) < \epsilon$. For every $n \geq N_0$, by (7), there exists $m \geq n$ such that $X(x_m, x) \leq \max\{\epsilon, \delta\}$. Hence, for all $n \geq N_0$, $X(x_n, x) \leq \max\{X(x_n, x_m), X(x_m, x)\} \leq \max\{\epsilon, \max\{\epsilon, \delta\}\} \leq \max\{\epsilon, \delta\}$, from which follows

$$X(\lim x_n, x) = \underleftarrow{\lim}\, X(x_n, x) = \inf_N \sup_{n \geq N} X(x_n, x) \leq \sup_{n \geq N_0} X(x_n, x) \leq \max\{\epsilon, \delta\}.$$

Since ϵ was arbitrary, $X(\lim_n x_n, x) \leq \delta$. $\square$

For a subset V of X we write $cl_S(V)$ for the closure of V in the generalized Scott topology, that is, $cl_S(V)$ is the smallest gS-closed set containing V. From the definition of limits we have that for any Cauchy sequence $(x_n)_n$ in V, $\lim x_n$ in $cl_S(V)$. The latter implies that if X is a gum with base B then B is *dense* in X, that is $cl_S(B) = X$. Indeed, $B \subseteq X$ implies $cl_S(B) \subseteq cl_S(X) = X$. For the converse we use the fact that every element of X is the limit of a Cauchy sequence in B.

Theorem 5.6 *Let X be an algebraic complete gum X with base B. For all subsets $V \subseteq X$, $cl_S(V) = \int_S \circ \rho_S(V)$.*

Proof: This theorem can be proved along the same lines as Proposition 3.6. It follows from the characterization (5) of $\int_S \circ \rho_S$ and the fact that the ϵ-balls of finite elements form a basis for the generalized Scott topology. $\square$

The specialization preorder on a complete gum X induced by its generalized Scott topology coincides with the preorder underlying X:

Proposition 5.7 *Let X be a complete gum X. For x and y in X, $x \leq_{\mathcal{O}_{gS}} y$ if and only if $x \leq_X y$.*

Proof: For any gS-open set V, if $x \in V$ and $X(x, y) = 0$, then $y \in V$. From this observation, the implication from right to left is clear. For the converse, suppose $X(x, y) \neq 0$. Then x in $X \setminus \{z \mid X(z, y) \leq 0\}$ but $y \notin X \setminus \{z \mid X(z, y) \leq 0\}$. Since, by Lemma 5.5, the set $X \setminus \{z \mid X(z, y) \leq 0\}$ is gS-open it follows that $x \not\leq_{\mathcal{O}_{gS}} y$. $\square$

Note that the specialization preorder is a *partial* order—or, equivalently the gS-topology is $\mathcal{T}_0$—if and only if X is an algebraic complete quasi ultrametric space.

This section is concluded with two observations relating limits and topological convergence. In the sequel we denote by $\mathcal{N}(x)$ the principal filter induced by x in X, that is, $\mathcal{N}(x) = \{V \in \mathcal{O}(X) \mid x \in V\}$. Similarly, for any sequence $(x_n)_n$ in X we denote by $\mathcal{N}((x_n)_n)$ the filter

$$\mathcal{N}((x_n)_n) = \{V \in \mathcal{O}(X) \mid \exists N\, \forall n \geq N,\ \ x_n \in V\}.$$

As usual, we say that $\mathcal{N}((x_n)_n)$ *converges* to x in X, denoted by $(x_n)_n \to_{\mathcal{O}} x$, if $\mathcal{N}(x) \subseteq \mathcal{N}((x_n)_n)$. Note that it is straightforward from the definition of convergence, that a sequence $(x_n)_n$ in an algebraic complete gum X converges (with respect to the gS-topology on X) to an element x in X, if and only if

$$\forall \epsilon > 0\, \forall a \in B,\ \ X(a, x) < \epsilon \ \Rightarrow\ \ (\exists N\, \forall n \geq N,\ X(a, x_n) < \epsilon).$$

In the next proposition we show that in a complete gum every Cauchy sequence is topologically convergent to its limit. However, not every convergent sequence is Cauchy: Provide the set $X = \{1, 2, \ldots, \omega\}$ with the distance function

$$X(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x = \omega \text{ and } y = n \\ 1 & \text{otherwise} \end{cases}$$

Then X is an algebraic complete quasi ultrametric space with X itself as base since there are no non-trivial Cauchy sequence. The sequence $(n)_n$ converges to ω but is not Cauchy.

Proposition 5.8 *Let X be a complete gum and $(x_n)_n$ a Cauchy sequence in X. Then for any y in X, $x_n \to_{\mathcal{O}_{gS}} y$ if and only if $y \leq_{\mathcal{O}_{gS}} \lim x_n$.*

Proof: Clearly $x_n \to_{\mathcal{O}_{gS}} \lim x_n$ and since gS-open sets are upper closed sets, $y \leq_{\mathcal{O}_{gS}} \lim x_n$ implies $x_n \to_{\mathcal{O}_{gS}} y$. For the converse assume $x_n \to_{\mathcal{O}_{gS}} y$ and suppose $y \not\leq_{\mathcal{O}_{gS}} \lim x_n$. Then there is a $\delta > 0$ such that $X(y, \lim x_n) \not\leq \delta$. So $y \in X \setminus \{z \mid X(z, \lim x_n) \leq \delta\}$, which is a gS-open set by Lemma 5.5. Since $x_n \to_{\mathcal{O}_{gS}} y$, we can choose N_0 so that for all $n \geq N_0$, $x_n \in X \setminus \{z \mid X(z, \lim x_n) \leq \delta\}$. But

$$0 = X(\lim x_n, \lim x_n) = \lim_{\leftarrow n} X(x_n, \lim x_n),$$

so there is an N_1 such that $X(x_n, \lim x_n) \leq \delta$ for all $n \geq N_1$. Taking $n \geq \max\{N_0, N_1\}$ gives a contradiction. Therefore $y \leq_{\mathcal{O}_{gS}} \lim x_n$. □

Recall that a function $f : X \to Y$ between two complete gum's is (metrically) continuous if $f(\lim x_n) = \lim f(x_n)$ for every Cauchy sequence $(x_n)_n$ in X. It is *topologically* continuous if the inverse image of a gS-open subset of Y is gS-open in X. The two notions are related as follows.

Proposition 5.9 *A non-expansive function $f : X \to Y$ between complete gum's is metrically continuous if and only if it is topologically continuous.*

Proof: Let $f : X \to Y$ be a non-expansive and metrically continuous function and let $V \subseteq Y$ be gS-open. We need to prove $f^{-1}(V)$ in $\mathcal{O}_{gS}(X)$ in order to conclude that f is topologically continuous. Indeed, for any Cauchy sequence $(x_n)_n$ in X we have

$$\begin{aligned}
\lim x_n \in f^{-1}(V) \quad &\Longleftrightarrow \quad f(\lim x_n) \in V \\
&\Longleftrightarrow \quad \lim f(x_n) \in V \quad [f \text{ metrically continuous}] \\
&\Rightarrow \quad \exists N\ \exists \epsilon > 0\ \forall n \geq N,\ B_\epsilon(f(x_n)) \subseteq V \quad [f \text{ non-expansive and} \\
&\qquad\quad (f(x_n))_n \text{ Cauchy sequence and } V \text{ gS-open}] \\
&\Rightarrow \quad \exists N\ \exists \epsilon > 0\ \forall n \geq N,\ B_\epsilon(x_n) \subseteq f^{-1}(V) \quad [f \text{ non-expansive}].
\end{aligned}$$

For the converse assume $f : X \to Y$ is topologically continuous and let $(x_n)_n$ be a Cauchy sequence in X. Because f is topologically continuous and, by Proposition 5.8, $x_n \to_{\mathcal{O}_{gS}} \lim x_n$, then $f(x_n) \to_{\mathcal{O}_{gS}} f(\lim x_n)$. But then, by Proposition 5.8 again, $f(\lim x_n) \leq_{\mathcal{O}_{gS}} \lim f(x_n)$. Since f is non-expansive,

$$\begin{aligned}
X(\lim f(x_n), f(\lim x_n)) &= \lim_{\leftarrow n} X(f(x_n), f(\lim x_n)) \\
&\leq \lim_{\leftarrow n} X(x_n, \lim x_n) \\
&= X(\lim x_n, \lim x_n) = 0.
\end{aligned}$$

Thus $\lim f(x_n) \leq_{\mathcal{O}_{gS}} f(x)$. By definition of limits we can then conclude that $f(\lim x_n)$ is a limit of $(f(x_n))_n$. □

6 Related work

Lawvere's work on generalized metric spaces as enriched categories ([Law73]) together with the more topological perspective of Smith ([Smy87]) have been our main source of inspiration for the present paper, which continues the work of Rutten ([Rut95]) and is part of the work of Bonsangue, van Breugel and Rutten ([BBR95]). Recently, we have been influenced also by the work of Flagg and Kopperman [FK95] and Wagner [Wag94]. Generalized ultrametric spaces are a special instance of Lawvere's $\mathcal{V}$-categories. The non-symmetric ultrametric for $[0,1]$ is also described and studied in his paper. The notion of forward Cauchy for a non-symmetric metric space is from [Smy87] as well as the notion of limit. A purely enriched-categorical definition of forward Cauchy sequences and of limits can be found in Wagner's [Wag94, Wag95]. The notion of finiteness and algebraicity for a generalized ultrametric spaces are from [Rut95]. Clearly we are working in the tradition of domain theory and metric spaces, for which Plotkin's [Plo83] and Engelking's [Eng89] have been our respective main source of information.

The comprehension schema as comparison between fuzzy predicates and subsets has been studied by Lawvere ([Law73]) and also by Kent ([Ken87]). The generalized Alexandroff topology by using specifying the open sets, has been studied by Flagg and Kopperman ([FK95]) in the context of $\mathcal{V}$-domains. The definition of the generalized Scott topology via the Yoneda embedding is new while its definition by specifying the basis is briefly mentioned in the conclusion of [Smy87]. According the terminology of the latter paper, our generalized Scott topology is the finest strongly appropriate topology for an algebraic complete quasi ultrametric space X whenever X is taken together with the standard quasi-uniformity. Recently, Flagg and Sünderhauf [FS95] have proved a quasi metric version of Hofmann's Theorem on the soberification of the generalized Alexandroff topology. It has as a corollary the fact that the generalized Scott topology on an algebraic complete quasi ultrametric space is sober. A generalized Scott topology is given also in [Wag95]. However, his notion of topology does not coincides with the standard set theoretical one: for example it does not coincides with the ordinary ϵ-ball topology in case of ordinary metric spaces.

Another important topological approach to quasi metric space, which need to be mentioned is that of, again, Smyth ([Smy91]) and Flagg and Kopperman ([FK95]). They consider quasi metric spaces equipped with the generalized Alexandroff topology. In order to reconcile metric spaces with complete partial orders they assign to partial orders a distance function which in general is not two-valued. Their approach to topology is much simpler than ours since much of the standard topological theorems for ordinary metric spaces can be adapted. The price to be paid for such simplicity is that this approach works only for a restricted class of spaces: they have to be spectral. Hence a full reconciliation between metric spaces and partial orders is not possible because only algebraic cpo's which are so called 2/3 SFP are spectral in their Scott topology. Also the work of Sünderhauf on quasi uniformities ([Sün94]) is along the same lines.

Other papers on reconciling complete partial orders with metric spaces are [WS81, CD85, Doi88, Mat94]. Different definitions of Cauchy sequence for non-symmetric metric spaces can be found in [RSV82].

References

[AR89] P. America and J.J.M.M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *Journal of Computer and System Sciences* 39(3):343–375, 1989.

[BBR95] M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten. Generalized ultrametric spaces: completion, topology, and powerdomains via the Yoneda embedding. Report CS-R9560, CWI, Amsterdam, 1995. FTP-available at `ftp.cwi.nl` as pub/CWIreports/AP/CS-R9560.ps.Z.

[BV95] J.W. de Bakker and E.P. de Vink. *Control flow semantics.* Foundations of Computing Series. The MIT Press, 1995. To appear.

[CD85] G. Comyn and M. Dauchet. Metric approximations in ordered domains. In M. Nivat and J.C. Reynolds, editors, *Algebraic Methods in Semantics*, pages 251–275. Cambridge University Press, 1985.

[Doi88] D. Doitchinov. On completeness of quasi-metric spaces. *Topology and its applications*, 30:127–148, 1988.

[Eng89] R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics.* Heldermann Verlag, Berlin, revised and completed edition, 1989.

[FK95] B. Flagg and R. Kopperman. Continuity spaces: reconciling domains and metric spaces. To appear, 1995.

[FS95] B. Flagg and P. Sünderhauf. Hofmann's theorem for $\mathcal{V}$-domains. Draft, 1995.

[GHK+80] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott. *A compendium of continuous lattices.* Springer-Verlag, 1980.

[Ken87] R.E. Kent. The metric closure powerspace construction. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of the 3rd MFPS*, volume 298 of *Lecture Notes in Computer Science*, pages 173–199, New Orleans, 1987. Springer-Verlag.

[Law73] F.W. Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, 43:135–166, 1973.

[Law86] F.W. Lawvere. Taking categories seriously. *Revista Colombiana de Matemáticas*, XX:147–178, 1986.

[Mat94] S.G. Matthews. Partial metric topology. In *Proceedings of the 8th Summer Conference on General Topology and Applications*, volume 728 of *Annals of the New York Academy of Sciences*, pages 183–197. New York, 1994.

[Plo83] G.D. Plotkin. Domains. University of Edinburgh, 1983.

[RSV82] I.L. Reilly, P.V. Subrahmanyam, and M.K. Vamanamurthy. Cauchy sequences in quasi-pseudo-metric spaces. *Monatshefte fur Mathematik*, 93:127–130, 1982.

[Rut95] J.J.M.M. Rutten. Elements of generalized ultrametric domain theory. Report CS-R9507, CWI, Amsterdam, 1995. FTP-available at `ftp.cwi.nl` as pub/CWIreports/AP/CS-R9507.ps.Z. To appear in *Theoretical Computer Science.*

[Smy87] M.B. Smyth. Quasi uniformities: reconciling domains with metric spaces. In M. Main, A. Melton, M. Mislove, and D. Schmidt, editors, *Proceedings of the 3rd MFPS*, volume 298 of *Lecture Notes in Computer Science*, pages 236–253, New Orleans, 1987. Springer-Verlag.

[Smy91] M.B. Smyth. Totally bounded spaces and compact ordered spaces as domains of computation. In G.M. Reed, A.W. Roscoe, and R.F. Wachter, editors, *Topology and Category Theory in Computer Science*, pages 207–229. Oxford University Press, 1991.

[Smy92] M.B. Smyth. Topology. In S. Abramsky, Dov M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 1, pages 641–761. Oxford University Press, 1992.

[SP82] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal of Computation* 11:761–783, 1982.

[Sün94] P. Sünderhauf. *Discrete approximation of spaces.* PhD thesis, Technische Hochschule Darmstadt, Germany, 1994.

[Wag94] K.R. Wagner. *Solving recursive domain equations with enriched categories.* PhD thesis, Carnegie Mellon University, Pittsburgh, July 1994. Technical report CMU-CS-94-159.

[Wag95] K.R. Wagner. Liminf convergence in ω-categories. Draft version, 1995.

[Win93] G. Winskel. *The formal semantics of programming languages, an introduction.* Foundations of Computing Series. The MIT Press, 1993.

[WS81] K. Weihrauch and U. Schreiber. Embedding metric spaces into cpo's. *Theoretical Computer Science*, 16(1):5–34, October 1981.

[Yon54] N. Yoneda. On the homology theory of modules. *Journal of the Faculty of Science,* University of Tokyo, Section I, 7(2):193–227, December 1954.