Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

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Computer Science/Department of Software Technology

CS-R9560 1995

Report CS-R9560
ISSN 0169-118X

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Abstract

Generalized ultrametric spaces are a common generalization of preorders and ordinary ultrametric spaces (Lawvere 1973, Rutten 1995). Combining Lawvere's (1973) enriched-categorical and Smyth' (1987, 1991) topological view on generalized (ultra)metric spaces, it is shown how to construct 1. completion, 2. topology, and 3. powerdomains for generalized ultrametric spaces. Restricted to the special cases of preorders and ordinary ultrametric spaces, these constructions yield, respectively: 1. chain completion and Cauchy completion; 2. the Alexandroff and the Scott topology, and the ϵ-ball topology; 3. lower, upper, and convex powerdomains, and the powerdomain of compact subsets. Interestingly, all constructions are formulated in terms of (an ultrametric version of) the Yoneda (1954) lemma.

AMS Subject Classification (1991): 68Q10, 68Q55
CR Subject Classification (1991): D.3.1, F.1.2, F.3.2
Keywords & Phrases: preorder, quasimetric, generalized metric, ultrametric, enriched category, Yoneda lemma, completion, topology, powerdomain.

*Supported by the Netherlands Organization for Scientific Research.

Contents

1 Overview

A generalized ultrametric space consists of a set X together with a distance function $X(-,-) : X \times X \to [0,1]$, satisfying $X(x,x) = 0$ and $X(x,z) \leq \max\{X(x,y),\ X(y,z)\}$, for all x, y, and z in X. The family of generalized ultrametric spaces contains all ordinary ultrametric spaces (for which the distance is moreover symmetric and different elements cannot have distance 0) as well as all preordered spaces (because a preorder relation is simply a discrete distance function mapping into the set $\{0,1\}$). Thus generalized ultrametric spaces provide a common generalization of both preordered spaces and ordinary ultrametric spaces, which is the main motivation for the present study.

Our sources of inspiration are the work of Lawvere on $\mathcal{V}$-categories and generalized metric spaces [Law73] and the work by Smyth on quasi metric spaces [Smy91], and we have been influenced by recent work of Flagg and Kopperman [FK95] and Wagner [Wag94]. The present paper continues earlier work [Rut95], in which some of the basic theory of generalized ultrametric spaces has been developed.

The guiding principle throughout is Lawvere's view of ultrametric spaces as $[0,1]$-*categories*, by which they are structures that are formally similar to (ordinary) categories. As a consequence, insights from category theory can be adapted to the world of ultrametric spaces. In particular, we shall give the ultrametric version of the famous *Yoneda Lemma*, which expresses, intuitively, that one may identify elements x of a generalized ultrametric space X with a description of the distances between the elements of X and x (formally, the function that maps any y in X to $X(y,x)$). This elementary insight (with an easy proof) will be shown to be of fundamental importance for the theory of generalized ultrametric spaces (and, a fortiori, both for order-theoretic and ultrametric domain theory as well). Notably it will give rise to

1. a definition of *completion* of generalized ultrametric spaces, generalizing both chain completion of preordered spaces and metric Cauchy completion;

2. a topology on generalized ultrametric spaces generalizing both the Scott topology for *arbitrary* preorders, and the metric ϵ-ball topology;

3. the definition and characterization of three powerdomains generalizing on the one hand the familiar lower, upper, and convex powerdomains from order-theory; and on the other hand the ultrametric powerdomain of compact subsets.

Our main motivation for considering generalized *ultra*metric spaces rather than generalized metric spaces (where one would have $X(x,z) \leq X(x,y) + X(y,z)$, for all x, y, and z in X) is the above mentioned fact that the distance induced by a preorder is indeed a generalized ultrametric. Generalized ultrametric spaces seem to arise moreover naturally in the semantics of programming languages, notably when dealing with transition systems (cf. [Rut95]). Finally, because of the strong triangle inequality ultrametric spaces are—from a computational point of view— better behaved than metric spaces. However, the results presented here on completion and topology of generalized ultrametric spaces apply equally well to generalized metric spaces. It is to be investigated whether the various characterization theorems for the powerdomains would still hold in the generalized metric case.

As mentioned above, generalized ultrametric spaces and the constructions that are given in the present paper both unify and generalize a substantial part of order-theoretic and ultrametric domain theory. Both disciplines play a central role in (to a large extent even came into existence because of) the semantics of programming languages (cf. recent textbooks such as [Win93] and [BV95], respectively). The use of generalized ultrametric spaces in semantics, or more precisely, in the study of transition systems, will be an important next step. The combination of results from [Rut95] (on domain equations) and the present paper is expected, for instance, to lead to domains that are suitable for simulation and bisimulation.

The paper is organized as follows. Sections 2 and 3 give the basic definitions and facts on generalized ultrametric spaces. After the Yoneda Lemma in Section 4, completion, topology and powerdomains are discussed in Sections 5, 6, and 7. Finally Section 8 discusses related work.

2 Generalized ultrametric spaces as $[0,1]$-categories

Generalized ultrametric spaces are introduced and shown to be $[0,1]$-categories in the sense of Lawvere. In order to see this, a brief recapitulation of Lawvere's enriched-categorical view of metric spaces is presented. The section concludes with a few basic definitions and properties to be used in the sequel. (The reader not familiar with category theory might want to skip the brief summary of enriched categories, and only look at the special case of generalized ultrametric spaces.)

A *generalized ultrametric space* (gum for short) is a set X together with a mapping

$$X(-,-) : X \times X \to [0,1]$$

which satisfies, for all x, y, and z in X,

1. $X(x,x) = 0$, and
2. $X(x,z) \leq \max\{X(x,y), X(y,z)\}$,

the so-called strong triangle inequality. The real number $X(x,y)$ will be called the distance from x to y. (Note that it is bounded by 1.) Examples of generalized ultrametric spaces are:

1. The set A^∞ of finite and infinite words over some given set A with distance function, for v and w in A^∞,

   $$A^\infty(v,w) = \begin{cases} 0 & \text{if } v \text{ is a prefix of } w \\ 2^{-n} & \text{otherwise,} \end{cases}$$

   where n is the length of the longest common prefix of v and w.

2. Any *preorder* $\langle P, \leq \rangle$ (satisfying for all p, q, and r in P, $p \leq p$, and if $p \leq q$ and $q \leq r$ then $p \leq r$) can be viewed as a generalized ultrametric space, by defining

   $$P(p,q) = \begin{cases} 0 & \text{if } p \leq q \\ 1 & \text{if } p \not\leq q. \end{cases}$$

   By a slight abuse of language, any gum stemming from a preorder in this way will itself be called a preorder.

3. The set $[0,1]$ with distance, for r and s in $[0,1]$,

   $$[0,1](r,s) = \begin{cases} 0 & \text{if } r \geq s \\ s & \text{if } r < s. \end{cases}$$

We briefly review Lawvere's [Law73] conception of metric spaces as $\mathcal{V}$-categories [EK65, Kel82]. Then we shall follow and further elaborate his approach for the special case of generalized ultrametric spaces, which will be shown to be $[0,1]$-categories. The main point is that, in general, many properties of $\mathcal{V}$-categories derive from the structure on the underlying category $\mathcal{V}$. In our case, therefore, many properties of generalized ultrametric spaces are determined by properties of $[0,1]$.

The starting point is a category $\mathcal{V}$ together with a functor

$$\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

which is symmetric and associative, and has a unit k. This defines a so-called symmetric monoidal structure on $\mathcal{V}$. The category $\mathcal{V}$ is required to be complete and cocomplete (i.e., all limits and colimits in $\mathcal{V}$ should exist), and its monoidal structure should be closed: that is, there exists an (internal hom) functor

$$Hom : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$$

such that for all a in $\mathcal{V}$, the functor $Hom(a, -)$ (mapping b in $\mathcal{V}$ to $Hom(a, b)$) is right adjoint to the functor $a \otimes -$ (which maps b in $\mathcal{V}$ to $a \otimes b$). A *$\mathcal{V}$-category*, or a category enriched in $\mathcal{V}$, is any set (more generally, class) X together with the assignment of an object $X(x, y)$ of $\mathcal{V}$ to every pair of elements $\langle x, y \rangle$ in X; the assignment of a $\mathcal{V}$-morphism

$$X(x, y) \otimes X(y, z) \to X(x, z)$$

to every triple $\langle x, y, z \rangle$ of elements in X; and the assignment of a $\mathcal{V}$-morphism

$$k \to X(x, x)$$

to every element x in X, satisfying a number of naturality conditions (omitted here since they are trivial in the particular case we are interested in).

For instance, the category of all sets is a (complete and cocomplete) symmetric monoidal closed category (where $\otimes$ is given by the Cartesian product, and any one element set is a unit). The corresponding $\mathcal{V}$-categories are just ordinary categories: $X(x, y)$ is given by the homset of all morphisms between two objects x and y in a category X, and the $\mathcal{V}$-morphisms that are required to exist are just mappings defining the composition of morphisms, and giving identity morphisms.

Generalized ultrametric spaces can now be seen to be $[0, 1]$-enriched categories as follows. First of all, $[0, 1]$ is shown to be a complete and cocomplete symmetric monoidal closed category. It is a category because it is a preorder (objects are the real numbers between 0 and 1; and for r and s in $[0, 1]$ there is a morphism from r to s if and only if $r \geq s$). It is complete and cocomplete: equalizers and coequalizers are trivial (because there is at most one arrow between any two elements of $[0, 1]$), the product $r \times s$ of two elements r and s in $[0, 1]$ is given by $\max\{r, s\}$, and their coproduct $r + s$ by $\min\{r, s\}$. More generally, products are given by sup, and coproducts are given by inf. The monoidal structure on $[0, 1]$ is given by

$$\max : [0, 1] \times [0, 1] \to [0, 1],$$

assigning to two real numbers their maximum, which is symmetric and associative, and for which 0 is the unit element. (Note that in this case the monoidal product is identical to the categorical product. In general this need not be the case.) Consider the following ('internal hom-') functor

$$[0, 1](-, -) : [0, 1]^{op} \times [0, 1] \to [0, 1],$$

which assigns to r and s in $[0, 1]$ the distance $[0, 1](r, s)$ as defined in the third example above. The following fundamental equivalence states that $[0, 1](t, -)$ is right-adjoint to $\max\{t, -\}$, for any r in $[0, 1]$:

Proposition 2.1 *For all r, s, and t in* $[0, 1]$,

$$\max\{t, s\} \geq r \text{ if and only if } s \geq [0, 1](t, r).$$

□

As a consequence, $[0, 1]$ is a (complete and cocomplete symmetric) monoidal closed category. (In fact, since the monoidal structure is given by the categorical product on $[0, 1]$, it is even Cartesian closed.)

The $[0, 1]$-categories are precisely the generalized ultrametric spaces introduced at the beginning of this section: sets X together with a mapping assigning to x and y in X an object, i.e., a real number $X(x, y)$ in $[0, 1]$. The existence of a $[0, 1]$-morphism from $X(x, y) \otimes X(y, z) = \max\{X(x, y), X(y, z)\}$ to $X(x, z)$ gives the second, and the existence of a morphism from $k = 0$ to $X(x, x)$ gives the first of the axioms for generalized ultrametric spaces.

As mentioned above, many constructions and properties of generalized ultrametric spaces are determined by the category $[0, 1]$. Important examples are the definitions of limit and completeness, presented in Section 3. Also the category of all gums, which is introduced next, inherits much of the structure of $[0, 1]$.

Let *Gums* be the category with generalized ultrametric spaces as objects, and *non-expansive* maps as arrows: i.e., mappings $f : X \to Y$ such that for all x and x' in X,

$$Y(f(x), f(x')) \leq X(x, x').$$

A map f is *isometric* if for all x and x' in X,

$$Y(f(x), f(x')) = X(x, x').$$

Two spaces X and Y are called isometric (isomorphic) if there exists an isometric bijection between them. The product $X \times Y$ of two gums X and Y is defined as the Cartesian product of their underlying sets, together with distance, for $\langle x, y\rangle$ and $\langle x', y'\rangle$ in $X \times Y$,

$$X \times Y(\langle x, y\rangle,\ \langle x', y'\rangle) = \max\{X(x, x'),\ Y(y, y')\}.$$

Note that this definition uses the product (max) of $[0, 1]$. The *exponent* of X and Y is defined by

$$Y^X = \{f : X \to Y \mid f \text{ is non-expansive }\},$$

with distance, for f and g in Y^X,

$$Y^X(f, g) = \sup\{Y(f(x), g(x)) \mid x \in X\}.$$

The fact that the category $[0, 1]$ is monoidal closed implies that the category *Gums* is monoidal closed as well: i.e., for all gums X, Y, and Z,

$$Z^{X \times Y} \cong (Z^Y)^X.$$

This section is concluded by a number of constructions and definitions for generalized ultrametric spaces that will be used in the sequel.

A generalized ultrametric space generally does not satisfy

3. if $X(x, y) = 0$ and $X(y, x) = 0$ then $x = y$,
4. $X(x, y) = X(y, x)$,

which are the additional conditions that hold for an *ordinary* ultrametric space. Therefore it is sometimes called a *pseudo-quasi* ultrametric space. A *quasi* ultrametric space is a gum which satisfies axioms 1, 2, and 3. Note that $[0, 1]$ is a quasi ultrametric space. A gum satisfying 1, 2, and 4 is called a *pseudo* ultrametric space.

The *opposite* X^{op} of a gum X is the set X with distance

$$X^{op}(x, x') = X(x', x).$$

With this definition, the distance function $X(-, -)$ can be described as a mapping

$$X(-, -) : X^{op} \times X \to [0, 1].$$

Using Proposition 2.1 one can easily show that $X(-, -)$ is non-expansive.

We saw that any preorder P induces a gum. (Note that a partial order induces a quasi ultrametric and that the non-expansive mappings between preorders are precisely the monotone maps.) Conversely, any gum X gives rise to a preorder $\langle X, \leq_X\rangle$, where $\leq_X$, called the *underlying* ordering of X, is given, for x and y in X, by

$$x \leq_X y \text{ if and only if } X(x, y) = 0.$$

Any (pseudo or quasi) ultrametric space is a fortiori a gum. Conversely, any gum X induces a pseudo ultrametric space X^s, the *symmetrization* of X, with distance

$$X^s(x, y) = \max\{X(x, y), X^{op}(x, y)\}.$$

For instance, the ordering that underlies A^∞ is the usual prefix ordering, and $(A^\infty)^s$ is the standard ultrametric on words. The generalized ultrametric on $[0, 1]$ induces the reverse of the usual ordering: for r and s in $[0, 1]$,

$r \leq_{[0,1]} s$ if and only if $s \leq r$;

and the symmetric version of $[0,1]$ is defined by

$$[0,1]^s(r,s) = \begin{cases} 0 & \text{if } r = s \\ \max\{r,s\} & \text{if } r \neq s. \end{cases}$$

Any gum X induces a quasi ultrametric space $[X]$ defined as follows. Let $\approx$ be the equivalence relation on X defined, for x and y in X, by

$$x \approx y \text{ iff } (X(x,y) = 0 \text{ and } X(y,x) = 0).$$

Let $[x]$ denote the equivalence class of x with respect to $\approx$, and $[X]$ the collection of all equivalence classes. Defining $[X]([x],[y]) = X(x,y)$ turns $[X]$ into a quasi ultrametric space. It has the following universal property: for any non-expansive mapping $f : X \to Y$ from X to a quasi ultrametric space Y there exists a unique non-expansive mapping $f' : [X] \to Y$ with $f'([x]) = f(x)$, for $x \in X$.

3 Cauchy sequences, limits, and completeness

Cauchy sequences are introduced. It is explained how such sequences look like in $[0,1]$, and how to define in $[0,1]$ the notion of metric limit. This will give rise to a definition of metric limit for arbitrary generalized ultrametric spaces. Furthermore the notions of completeness, finiteness, and algebraicity are introduced.

A sequence $(x_n)_n$ in a generalized ultrametric space X is *forward-Cauchy* if

$$\forall \epsilon > 0 \ \exists N \ \forall n \geq N, \ \ X(x_n, x_{n+1}) \leq \epsilon.$$

Note that this is equivalent to the more familiar condition:

$$\forall \epsilon > 0 \ \exists N \ \forall n \geq m \geq N, \ \ X(x_m, x_n) \leq \epsilon,$$

because of the strong triangle inequality. Since our metrics need not be symmetric, the following variation exists: a sequence $(x_n)_n$ is *backward-Cauchy* if

$$\forall \epsilon > 0 \ \exists N \ \forall n \geq N, \ \ X(x_{n+1}, x_n) \leq \epsilon.$$

If X is an ordinary ultrametric space then forward-Cauchy and backward-Cauchy both mean Cauchy in the usual sense. And if X is a preorder then Cauchy sequences are eventually increasing: there exists an N such that for all $n \geq N$, $x_n \leq x_{n+1}$. (Increasing sequences in a preorder are also called *chains*.) Similarly backward-Cauchy sequences are eventually decreasing.

Cauchy sequences in $[0,1]$, with the generalized ultrametric of Section 2, are particularly simple: every forward-Cauchy sequence either converges to 0 or is eventually decreasing; dually, every backward-Cauchy sequence either converges to 0 or is eventually increasing.

Proposition 3.1 *A sequence $(r_n)_n$ in $[0,1]$ is forward-Cauchy if and only if*

either: $\forall \epsilon > 0 \ \exists N \ \forall n \geq N, \ \ r_n \leq \epsilon,$ *or:* $\exists N \ \forall n \geq N, \ \ r_n \geq r_{n+1}.$

Dually, it is backward-Cauchy if and only if

either: $\forall \epsilon > 0 \ \exists N \ \forall n \geq N, \ \ r_n \leq \epsilon,$ *or:* $\exists N \ \forall n \geq N, \ \ r_n \leq r_{n+1}.$

Proof: We prove only the first statement, the second being dual. Sequences that converge to 0 or that are eventually decreasing are easily seen to be forward-Cauchy. Conversely, let $(r_n)_n$ be forward-Cauchy in $[0,1]$. Suppose there exists $\epsilon > 0$ such that

$$\forall N \ \exists n \geq N, \ \ r_n > \epsilon.$$

We claim that there exists an N such that *for all* $n \geq N$, $r_n > \epsilon$; for suppose not:

$$\forall N\ \exists n \geq N,\quad r_n \leq \epsilon.$$

Because $(r_n)_n$ is forward-Cauchy, there exists M such that for all $m \geq M$, $[0,1](r_m, r_{m+1}) \leq \epsilon$. Consider $n_1 \geq M$ with $r_{n_1} \leq \epsilon$, and consider $n_2 \geq n_1$ with $r_{n_2} > \epsilon$. Then

$$\begin{array}{rcll} \epsilon & < & r_{n_2} & \\ & = & [0,1](r_{n_1}, r_{n_2}) & \text{[definition distance on } [0,1]\text{]} \\ & \leq & \epsilon, & \end{array}$$

a contradiction. Therefore let N be such that for all $n \geq N$, $r_n > \epsilon$. Let $M \geq N$ such that for all $m \geq M$, $[0,1](r_m, r_{m+1}) \leq \epsilon$, which is equivalent to $r_{m+1} \leq \max\{\epsilon, r_m\}$ by Proposition 2.1. Because $r_m > \epsilon$, for all $m \geq M$, this implies $r_{m+1} \leq r_m$. □

Because Cauchy sequences in $[0,1]$ are that simple, the following definitions are easy as well: the *forward-limit* of a forward-Cauchy sequence $(r_n)_n$ in $[0,1]$ is given by

$$\lim_{\rightarrow} r_n = \sup_n \inf_{k \geq n} r_k.$$

Dually, the *backward-limit* of a backward-Cauchy sequence $(r_n)_n$ in $[0,1]$ is

$$\lim_{\leftarrow} r_n = \inf_n \sup_{k \geq n} r_k.$$

These numbers are what one intuitively would consider as metric limits of Cauchy sequences. If $[0,1]$ is taken with the standard Euclidian metric: $d(r, r') = |r - r'|$, for r and r' in $[0,1]$, then both forward-Cauchy and backward-Cauchy sequences are Cauchy with respect to d, and the forward-limit and backward-limit defined above coincide with the usual notion of limit with respect to d.

The following proposition shows how forward-limits and backward-limits in $[0,1]$ are related (cf. [Wag95]).

Proposition 3.2 *For a forward-Cauchy sequence $(r_n)_n$ in $[0,1]$, and all r in $[0,1]$,*

$$[0,1](\lim_{\rightarrow} r_n, r) = \lim_{\leftarrow} [0,1](r_n, r).$$

For a backward-Cauchy sequence $(r_n)_n$ in $[0,1]$, and all r in $[0,1]$,

$$[0,1](r, \lim_{\leftarrow} r_n) = \lim_{\leftarrow} [0,1](r, r_n).$$

□

A proof follows easily from the following elementary facts:

Lemma 3.3 *For all $V \subseteq [0,1]$ and r in $[0,1]$,*

1. $$[0,1](\inf V, r) = \sup_{v \in V} [0,1](v, r);$$

2. $$[0,1](r, \sup V) = \sup_{v \in V} [0,1](r, v);$$

3. $$[0,1](r, \inf V) \leq \inf_{v \in V} [0,1](r, v).$$

If V is finite then the latter inequality is in fact an equality. □

Forward-limits in an *arbitrary* generalized ultrametric space X can now be defined in terms of backward-limits in $[0,1]$: an element x is a *forward-limit* of a forward-Cauchy sequence $(x_n)_n$ in X,

$$x = \lim_{\rightarrow} x_n \text{ iff } \forall y \in X,\quad X(x, y) = \lim_{\leftarrow} X(x_n, y).$$

This is well defined because of the following.

Proposition 3.4 *Let $(x_n)_n$ be a forward Cauchy sequence in X. Let $x \in X$.*

1. *The sequence $(X(x, x_n))_n$ is forward Cauchy in $[0,1]$.*
2. *The sequence $(X(x_n, x))_n$ is backward Cauchy in $[0,1]$.*

Note that our earlier definition of the forward-limit of forward-Cauchy sequences in $[0,1]$ is consistent with this definition for arbitrary gums: this follows from the first statement of Proposition 3.2.

For ordinary ultrametric spaces, the above defines the usual notion of limit:

$$x = \lim_{\rightarrow} x_n \text{ if and only if } \forall \epsilon > 0\ \exists N\ \forall n \geq N,\ \ X(x_n, x) < \epsilon.$$

If X is a preorder and $(x_n)_n$ is a chain in X then

$$x = \lim_{\rightarrow} x_n \text{ if and only if } \forall y \in X,\ \ x \leq_X y \Leftrightarrow \forall n \geq 0,\ x_n \leq_X y,$$

i.e., $x = \bigsqcup x_n$, the least upperbound of the chain $(x_n)_n$.

One could also consider backward-limits for arbitrary gums. Since these will not play a role in the rest of this paper, this is omitted. For simplicity, we shall use *Cauchy* instead of forward-Cauchy. Similarly, we shall write

$$\lim x_n \text{ rather than } \lim_{\rightarrow} x_n.$$

Note that subsequences of a Cauchy sequence are Cauchy again. If a Cauchy sequence has a limit x, then all its subsequences have limit x as well. Cauchy sequences may have more than one limit. All limits have distance 0, however. As a consequence, limits are unique in quasi ultrametric spaces.

The following fact will be useful in the future:

Proposition 3.5 *Let $(x_n)_n$ be a forward Cauchy sequence in X. Let $x \in X$.*

$$X(x, \lim_n x_n) \leq \lim_n X(x, x_n).$$

Proof The inequality follows from

$$\begin{aligned}
&[0,1](\lim_n X(x, x_n),\ X(x, \lim_n x_n)) \\
&= \lim_{\leftarrow}{}_n [0,1](X(x, x_n),\ X(x, \lim_n x_n)) \\
&\leq \lim_{\leftarrow}{}_n X(x_n, \lim_n x_n) \quad \text{[the mapping } X(x, -) : X \rightarrow [0,1] \text{ is non-expansive]} \\
&= X(\lim_n x_n,\ \lim_n x_n) \\
&= 0.
\end{aligned}$$

□

A generalized ultrametric space X is *complete* if every Cauchy sequence in X has a limit. A subset $V \subseteq X$ is complete if every Cauchy sequence in V has a limit in V. For instance, $[0,1]$ is complete. If X is a partial order completeness means that X is a complete partial order, cpo for short: all ω-chains have a least upperbound. For ordinary ultrametric spaces this definition of completeness is the usual one. There is the following fact (cf. Theorem 6.5 of [Rut95]).

Proposition 3.6 *Let X and Y be generalized ultrametric spaces. If Y is complete then Y^X is complete. Moreover, limits are pointwise: let $(f_n)_n$ be a Cauchy sequence in Y^X and f an element in Y^X. Then $\lim f_n = f$ if and only if for all $x \in X$, $\lim f_n(x) = f(x)$. Furthermore, if Y is a quasi ultrametric space then Y^X is a quasi ultrametric space as well.* □

A mapping $f : X \to Y$ between gums X and Y is *continuous* if it preserves limits: if $x = \lim x_n$ in X then $f(x) = \lim f(x_n)$ in Y. For ordinary ultrametric spaces, this is the usual definition. For preorders it means preservation of least upperbounds of ω-chains.

An element a in a generalized ultrametric space X is *finite* if the mapping

$$X(a, -) : X \to [0,1], \quad x \mapsto X(a, x)$$

is continuous. (So for finite elements, the inequality in Proposition 3.5 actually is an equality.) If X is a preorder this means that for any chain $(x_n)_n$ in X,

$$X(a, \bigsqcup x_n) = \underleftarrow{\lim} X(a, x_n),$$

or, equivalently,

$$a \leq_X \bigsqcup x_n \text{ iff } \exists n, \quad a \leq_X x_n,$$

which is the usual definition for ordered spaces. If X is an ordinary ultrametric space then $X(a, -)$ is continuous for any a in X, hence all elements are finite.

A *basis* for a generalized ultrametric space X is a subset $B \subseteq X$ consisting of finite elements such that every element x in X is the limit $x = \lim a_n$ of a Cauchy sequence $(a_n)_n$ of elements in B. A gum X is *algebraic* if there exists a basis for X. Note that such a basis is in general not unique. If X is algebraic then the collection B_X of all finite elements of X is the largest basis. Further note that algebraic does not imply complete. (Take any ordinary ultrametric space which is not complete.) If there exists a countable basis then X is ω*-algebraic*. For instance, the generalized ultrametric space A^∞ from Section 2 is algebraic with basis A^*, the set of all finite words over A. If A is countable then A^∞ is ω-algebraic.

4 The Yoneda Lemma

The following lemma turns out to be of great importance for the theory of generalized ultrametric spaces. It is the $[0,1]$-categorical version of the famous *Yoneda Lemma* [Yon54] from category theory. We shall see in the subsequent sections that it gives rise to elegant definitions and characterizations of completion, topology, and powerdomains. A general proof of the Yoneda Lemma for arbitrary $\mathcal{V}$-categories can be found in [Kel82]. For generalized metric spaces, it is proved in [Law86].

The following notation will be used throughout the rest of this paper:

$$\hat{X} = [0,1]^{X^{op}},$$

i.e., the set of all non-expansive functions from X^{op} to $[0,1]$.

Lemma 4.1 (Yoneda Lemma) *Let X be a generalized ultrametric space. For any $x \in X$ let*

$$X(-, x) : X^{op} \to [0,1], \quad y \mapsto X(y, x).$$

This function is non-expansive and hence an element of $\hat{X}$. For any other element ϕ in $\hat{X}$, $\hat{X}(X(-, x),\ \phi) = \phi(x)$.

Proof: Because $X(-,-) : X^{op} \times X \to [0,1]$ is non-expansive, so is $X(-, x)$, for any x in X. Now let $\phi \in \hat{X}$. On the one hand,

$$\begin{aligned} \phi(x) &= [0,1](X(x,x),\ \phi(x)) \\ &\leq \sup_{y \in X} [0,1](X(y,x),\ \phi(y)) \\ &= \hat{X}(X(-,x),\ \phi). \end{aligned}$$

On the other hand, non-expansiveness of ϕ gives, for any y in X,

$$[0,1](\phi(x),\ \phi(y)) \leq X^{op}(x,y) = X(y,x),$$

which is equivalent by Proposition 2.1 to $[0,1](X(y,x),\ \phi(y)) \leq \phi(x)$. □

The following corollary is immediate.

Corollary 4.2 *The* **Yoneda embedding** $\mathbf{y} : X \to \hat{X}$, *defined for x in X by* $\mathbf{y}(x) = X(-,x)$ *is isometric: for all x and x' in X,*

$$X(x,x') = \hat{X}(\mathbf{y}(x), \mathbf{y}(x')).$$

□

The following fact will be of use when defining completion.

Lemma 4.3 *For any x in X, $\mathbf{y}(x)$ is finite in $\hat{X}$.*

Proof: We have to show that $\hat{X}(\mathbf{y}(x), -) : \hat{X} \to [0,1]$ is continuous: for any Cauchy sequence $(\phi_n)_n$ in $\hat{X}$,

$$\begin{aligned}\hat{X}(\mathbf{y}(x), \lim \phi_n) &= (\lim \phi_n)(x) \quad \text{[the Yoneda Lemma]}\\ &= \lim \phi_n(x) \quad \text{[Proposition 3.6]}\\ &= \lim \hat{X}(\mathbf{y}(x), \phi_n) \quad \text{[the Yoneda Lemma].}\end{aligned}$$

□

5 Completion via Yoneda

The completion of generalized ultrametric spaces is defined by means of the Yoneda embedding. It yields for ordinary ultrametric spaces Hausdorff's standard Cauchy completion (as introduced in [Hau14]), for preorders the chain completion, and for quasi ultrametric spaces a completion given by Smyth (see page 214 of [Smy91]).

Let X be a generalized ultrametric space. Because $[0,1]$ is a complete quasi ultrametric space (cf. Section 2 and 3), it follows from Proposition 3.6 that $\widehat{X}$ is a complete quasi ultrametric space as well. According to Corollary 4.2, the Yoneda embedding $\mathbf{y}$ isometrically embeds X in $\widehat{X}$. The completion of X can now be defined as the smallest complete subspace of $\widehat{X}$ which contains the $\mathbf{y}$-image of X.

Definition 5.1 The *completion* of a generalized ultrametric space X is defined by

$$\bar{X} = \bigcap \{\, V \text{ is a complete subspace of } \widehat{X} \mid \mathbf{y}(X) \subseteq V \,\}.$$

The collection of which the intersection is taken is nonempty, since it contains $\widehat{X}$. Because $\bar{X}$ is a complete subspace of the complete quasi ultrametric space $\widehat{X}$, also $\bar{X}$ is a complete quasi ultrametric space, and, as a consequence, for any Cauchy sequence in $\bar{X}$, its limits in $\bar{X}$ and $\hat{X}$ coincide.

As with preorders, completion is not idempotent, that is, the completion of the completion of X is in general not isomorphic to the completion of X. An interesting question is to characterize the family of generalized ultrametric spaces for which completion is idempotent (it contains at least all ordinary ultrametric spaces).

Completion for ordinary (ultra)metric spaces is usually defined by means of (equivalence classes of) Cauchy sequences. The same applies to countable preorders: there the most common form of completion, ideal completion, is isomorphic to chain completion, and we have seen that chains are (special cases of) Cauchy sequences. It will be shown next that the completion introduced above can be expressed in terms of Cauchy sequences as well. This will at the same time enable us to prove its equivalence with the definition of the completion of quasi metric spaces by Smyth.

Note that a sequence $(x_n)_n$ is Cauchy in a generalized ultrametric space X if and only if $(\mathbf{y}(x_n))_n$ is Cauchy in $\widehat{X}$, because the Yoneda embedding $\mathbf{y}$ is isometric. This is used in the following.

Proposition 5.2 *For any generalized ultrametric space X,*

$$\bar{X} = \{\lim_n \mathbf{y}(x_n) \mid (x_n)_n \text{ is a Cauchy sequence in } X\}.$$

Proof The inclusion from right to left is immediate from the fact that the set on the right is contained in any complete subspace V of $\widehat{X}$ which contains $\mathbf{y}(X)$. The reverse inclusion follows from the fact that the set on the right contains $\mathbf{y}(X)$, which is trivial, and the fact that it is a complete subspace of $\widehat{X}$: this is a consequence of Lemma 4.3 and Proposition B.3 in the appendix. $\square$

The elements of $\bar{X}$ can be seen to represent equivalence classes of Cauchy sequences. To this end, let $CS(X)$ denote the set of all Cauchy sequences in X, and let $\lambda : CS(X) \to \bar{X}$ map a Cauchy sequence $(v_n)_n$ in X to $\lim_n \mathbf{y}(v_n)$. This mapping induces a generalized ultrametric structure on $CS(X)$ by putting, for Cauchy sequences $(v_n)_n$ and $(w_n)_n$,

$$CS(X)((v_n)_n,\ (w_n)_n) = \bar{X}(\lambda((v_n)_n), \lambda((w_n)_n)).$$

This metric can be characterized as follows:

$$\begin{aligned}
& CS(X)((v_n)_n,\ (w_n)_n) \\
&= \bar{X}(\lambda((v_n)_n), \lambda((w_n)_n)) \\
&= \bar{X}(\lim_n \mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)) \\
&= \widehat{X}(\lim_n \mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)) \\
&= \underleftarrow{\lim}_n \widehat{X}(\mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)) \\
&= \underleftarrow{\lim}_n \lim_m \widehat{X}(\mathbf{y}(v_n), \mathbf{y}(w_m)) \quad [\mathbf{y}(v_n) \text{ is finite in } \widehat{X}] \\
&= \underleftarrow{\lim}_n \lim_m X(v_n, w_m) \quad [\mathbf{y} \text{ is isometric}].
\end{aligned}$$

The latter formula is what Smyth has used for a definition of the distance between Cauchy sequences of quasi metric spaces. In his approach, the completion of a quasi metric space is defined as $[CS(X)]$, which is the quasi metric space obtained from $CS(X)$ by identifying all Cauchy sequences with distance 0 in both directions (cf. Section 2). Such sequences can be considered to represent the same limit. Both ways of defining completion are equivalent:

Proposition 5.3 *For any generalized ultrametric space X, $\bar{X} \cong [CS(X)]$.*

Proof Because $\bar{X}$ is a quasi ultrametric space, the mapping $\lambda : CS(X) \to \bar{X}$ induces a non-expansive mapping $\lambda' : [CS(X)] \to \bar{X}$ (cf. Section 2). Because λ is isometric by the definition of the metric on $CS(X)$, λ' is injective. Because λ is surjective by Proposition 5.2, λ' is also surjective. $\square$

A corollary of this theorem is that the completion of generalized ultrametric spaces generalizes Cauchy completion of ordinary ultrametric spaces and chain completion of preorders.

Recall that the category *Gums* has generalized ultrametric spaces as objects and non-expansive functions as arrows. Let *Acq* be the category with algebraic complete quasi ultrametric spaces as objects, and with non-expansive and continuous functions as arrows. We will show that completion can be extended to a functor from *Gums* to *Acq*, which is a left adjoint to the forgetful functor from *Acq* to *Gums*. First of all, the completion of a generalized ultrametric space X is an object in *Acq*:

Theorem 5.4 *For any generalized ultrametric space X, $\bar{X}$ is an algebraic complete quasi ultrametric space.*

Proof Since $\bar{X}$ is a complete subspace of the complete quasi ultrametric space $\widehat{X}$, also $\bar{X}$ is a complete quasi ultrametric space. Because all elements of $\mathbf{y}(X)$ are finite in $\widehat{X}$ according to Lemma 4.3, they are also finite in $\bar{X}$. From Proposition 5.2 we can conclude that every element of $\bar{X}$ is the limit of a Cauchy sequence in $\mathbf{y}(X)$. Consequently $\bar{X}$ is algebraic. $\square$

The next theorem is the key to the extension of completion to a functor. It says that completion is a so-called free construction:

Theorem 5.5 *For any complete quasi ultrametric space Y and non-expansive function $f : X \to Y$ there exists a unique non-expansive and continuous function $f^{\#} : \bar{X} \to Y$ such that $f^{\#} \circ \mathbf{y} = f$.*

$$\begin{tikzcd} X \arrow[r, "\mathbf{y}"] \arrow[dr, "f"'] & \bar{X} \arrow[d, dotted, "f^{\#}"] \\ & Y \end{tikzcd}$$

Proof For all Cauchy sequences $(v_n)_n$ and $(w_m)_m$ in X,

$$\begin{aligned} & Y(\lim_n f(v_n), \lim_m f(w_m)) \\ = \;& \varprojlim_n Y(f(v_n), \lim_m f(w_m)) \\ \leq \;& \varprojlim_n \lim_m Y(f(v_n), f(w_m)) \quad [\text{Proposition 3.5}] \\ \leq \;& \varprojlim_n \lim_m X(v_n, w_m) \quad [f \text{ is non-expansive}] \\ = \;& \varprojlim_n \lim_m \widehat{X}(\mathbf{y}(v_n), \mathbf{y}(w_m)) \quad [\mathbf{y} \text{ is isometric}] \\ = \;& \varprojlim_n \widehat{X}(\mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)) \quad [\mathbf{y}(v_n) \text{ is finite in } \widehat{X}] \\ = \;& \widehat{X}(\lim_n \mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)). \end{aligned}$$

Consequently,

$$\begin{aligned} & \lim_n \mathbf{y}(v_n) = \lim_m \mathbf{y}(w_m) \\ \Rightarrow \;& \widehat{X}(\lim_n \mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)) = 0 \wedge \widehat{X}(\lim_m \mathbf{y}(w_m), \lim_n \mathbf{y}(v_n)) = 0 \\ \Rightarrow \;& Y(\lim_n f(v_n), \lim_m f(w_m)) = 0 \wedge Y(\lim_m f(w_m), \lim_n f(v_n)) = 0 \\ \Rightarrow \;& \lim_n f(v_n) = \lim_m f(w_m). \end{aligned}$$

According to Proposition 5.2, for all $\bar{x} \in \bar{X}$, there exists a Cauchy sequence $(x_n)_n$ in X, such that $\bar{x} = \lim_n \mathbf{y}(x_n)$. Since f is non-expansive, the sequence $(f(x_n))_n$ is also Cauchy. Because Y is a complete quasi ultrametric space, $\lim_n f(x_n)$ exists. Hence, we can define $f^{\#} : \bar{X} \to Y$ by

$$f^{\#}(\lim_n \mathbf{y}(x_n)) = \lim_n f(x_n).$$

Since, for all Cauchy sequences $(v_n)_n$ and $(w_m)_m$ in X,

$$\begin{aligned} & Y(f^{\#}(\lim_n \mathbf{y}(v_n)), f^{\#}(\lim_m \mathbf{y}(w_m))) \\ = \;& Y(\lim_n f(v_n), \lim_n f(w_m)) \\ = \;& \widehat{X}(\lim_n \mathbf{y}(v_n), \lim_m \mathbf{y}(w_m)) \quad [\text{see above}] \end{aligned}$$

the function $f^{\#}$ is non-expansive.

Next we prove that $f^{\#}$ is continuous. Let $(\bar{x}_n)_n$ be a Cauchy sequence in $\bar{X}$. Without loss of generality we can assume that

$$\forall n : \bar{X}(\bar{x}_n, \bar{x}_{n+1}) \leq \tfrac{1}{3n}. \tag{1}$$

According to Proposition 5.2, we have that

$$\bar{X} = \{ \lim_n \mathbf{y}(x_n) \mid (x_n)_n \text{ is a Cauchy sequence in } X \}.$$

Because $\mathbf{y}(X)$ is a subspace of the complete quasi ultrametric space $\widehat{X}$, and all elements of $\mathbf{y}(X)$ are finite in $\widehat{X}$ according to Lemma 4.3, we can conclude from Lemma B.1 and B.2 that there exist Cauchy sequences $(w_n^m)_m$ in $\mathbf{y}\,(X)$ satisfying

$$\begin{aligned}
&\forall m : \forall n : \mathbf{y}\,(X)\,(w_n^m, w_{n+1}^m) \leq \tfrac{1}{n},\\
&\forall n : \forall m : \mathbf{y}\,(X)\,(w_n^m, w_n^{m+1}) \leq \tfrac{1}{m},\\
&\forall n : \lim_m w_n^m = \bar{x}_n,\\
&\lim_k w_k^k = \lim_n \bar{x}_n.
\end{aligned}$$

Since $\mathbf{y}$ is isometric, there exist Cauchy sequences $(x_n^m)_m$ in X satisfying

$$\forall m : \forall n : X\,(x_n^m, x_{n+1}^m) \leq \tfrac{1}{n}, \tag{2}$$
$$\forall n : \forall m : X\,(x_n^m, x_n^{m+1}) \leq \tfrac{1}{m}, \tag{3}$$
$$\forall n : \lim_m \mathbf{y}\,(x_n^m) = \bar{x}_n, \tag{4}$$
$$\lim_k \mathbf{y}\,(x_k^k) = \lim_n \bar{x}_n. \tag{5}$$

As we have seen above, $f^\#$ is non-expansive. Consequently, $(f^\#\,(\bar{x}_n))_n$ is a Cauchy sequence in Y. From (1) we can conclude that

$$\forall n : Y\,(f^\#\,(\bar{x}_n), f^\#\,(\bar{x}_{n+1})) \leq \tfrac{1}{3n}\ . \tag{6}$$

Since f is non-expansive, we can derive from (2) and (3) that

$$\forall m : \forall n : Y\,(f\,(x_n^m), f\,(x_{n+1}^m)) \leq \tfrac{1}{n}, \tag{7}$$
$$\forall n : \forall m : Y\,(f\,(x_n^m), f\,(x_n^{m+1})) \leq \tfrac{1}{m}. \tag{8}$$

From (4) we can deduce that

$$\forall n : \lim_m f\,(x_n^m) = f^\#\,(\bar{x}_n). \tag{9}$$

Since Y is a complete quasi ultrametric space, it follows from (6), (7), (8), (9), and Lemma B.2 that the sequence $(f\,(x_k^k))_k$ is Cauchy and

$$\lim_k f\,(x_k^k) = \lim_n f^\#\,(\bar{x}_n).$$

From (5) we can derive that

$$f^\#\,(\lim_n \bar{x}_n) = \lim_k f\,(x_k^k).$$

Hence $f^\#$ is continuous.

Let $g : \bar{X} \to Y$ be a non-expansive and continuous function such that $g \circ \mathbf{y} = f$. For all Cauchy sequences $(x_n)_n$ in X,

$$\begin{aligned}
&g\,(\lim_n \mathbf{y}\,(x_n))\\
&\quad = \quad \lim_n g\,(\mathbf{y}\,(x_n)) \quad [g \text{ is continuous}]\\
&\quad = \quad \lim_n f\,(x_n) \quad [g \circ \mathbf{y} = f]\\
&\quad = \quad f^\#\,(\lim_n \mathbf{y}\,(x_n)).
\end{aligned}$$

This proves the unicity of $f^\#$. □

Completion can be extended to a functor $(\bar{\ })$: *Gums* $\to$ *Acq*, by defining its action on arrows in *Gums* in the following standard way: for generalized ultrametric spaces X and Y and a non-expansive mapping $f : X \to Y$, let $\bar{f} : \bar{X} \to \bar{Y}$ be defined by $\bar{f} = (\mathbf{y} \circ f)^\#$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathbf{y}"'] & Y \arrow[d, "\mathbf{y}"] \\
\bar{X} \arrow[r, dashed, "(\mathbf{y}\circ f)^{\#}"'] & \bar{Y}
\end{tikzcd}$$

According to Theorem 5.5, the function $\bar{f}$ is non-expansive and continuous, and hence an arrow in *Acq*. One can easily verify that we have extended completion to a functor. It is an immediate consequence of Theorem 5.5 that it is left adjoint to the forgetful functor from *Acq* to *Gums* (cf. Chapter 4 of [ML71]). The Yoneda embedding $\mathbf{y}$ is the unit of the adjunction.

For every complete quasi ultrametric space X with basis A, $X \cong \bar{A}$. More generally:

Theorem 5.6 *Let X be a complete quasi ultrametric space. Let $A \subseteq X$. Then the following three conditions are equivalent.*

1. *A is a basis for X.*

2. *The function $\mathbf{y}_A : X \to \widehat{A}$ defined, for $x \in X$, by*

$$\mathbf{y}_A\,(x) = \lambda a \in A\,.\,X\,(a, x)$$

(i.e., the restriction of $\mathbf{y}(x) \in \hat{X}$ to A) is isometric and continuous.

3. *The inclusion function $i : A \to X$ induces an isomorphism $i^{\#} : \bar{A} \to X$.*

Proof

1. $\Rightarrow$ 2. According to Corollary 4.2, $\mathbf{y}$ is isometric. Consequently, $\mathbf{y}_A$ is non-expansive. Because, for all Cauchy sequences $(x_n)_n$ in X,

$$\begin{array}{ll}
\multicolumn{2}{l}{\lim_n \mathbf{y}_A\,(x_n)} \\
= & \lim_n \lambda a \in A\,.\,X\,(a, x_n) \\
= & \lambda a \in A\,.\,\lim_n X\,(a, x_n) \quad [\text{Proposition 3.6}] \\
= & \lambda a \in A\,.\,X\,(a, \lim_n x_n) \quad [a \text{ is finite in } X] \\
= & \mathbf{y}_A\,(\lim_n x_n),
\end{array}$$

$\mathbf{y}_A$ is continuous. Consider the following diagram:

$$\begin{tikzcd}
& A \arrow[d, hook, "i"] \arrow[ddl, "\mathbf{y}"'] \arrow[ddr, "\mathbf{y}"] & \\
& X \arrow[dl, "\mathbf{y}_A"'] & \\
\widehat{A} & & \bar{A} \arrow[ll, hook', "j"] \arrow[ul, "i^{\#}"']
\end{tikzcd}$$

where j is the inclusion of $\bar{A}$ in $\hat{A}$. One can easily verify that $\mathbf{y}_A \circ i^{\#} \circ \mathbf{y} = \mathbf{y}$ and $j \circ \mathbf{y} = \mathbf{y}$. Therefore by Theorem 5.5,

$$\mathbf{y}_A \circ i^{\#} = j. \tag{10}$$

Since A is a basis for X, $i^{\#}$ is surjective. Because $i^{\#}$ is furthermore non-expansive and j is isometric, $\mathbf{y}_A$ is isometric.

2. $\Rightarrow$ 3. For all Cauchy sequences $(a_n)_n$ in A,

$$\begin{array}{cl} & (\mathbf{y}_A \circ i^\#)\,(\lim_n \mathbf{y}\,(a_n)) \\ = & \mathbf{y}_A\,(\lim_n i^\# \circ \mathbf{y}\,(a_n)) \quad [i^\# \text{ is continuous}] \\ = & \mathbf{y}_A\,(\lim_n i\,(a_n)) \\ = & \lim_n \mathbf{y}_A \circ i\,(a_n) \quad [\mathbf{y}_A \text{ is continuous}] \\ = & \lim_n \mathbf{y}\,(a_n), \end{array}$$

from which (10) follows. Thus $\mathbf{y}_A$ actually maps into $\bar{A}$. Because $\mathbf{y}_A$ is isometric it is injective. As a consequence, $i^\# \circ \mathbf{y}_A = 1_X$ follows from

$$\mathbf{y}_A \circ (i^\# \circ \mathbf{y}_A) = (\mathbf{y}_A \circ i^\#) \circ \mathbf{y}_A = \mathbf{y}_A = \mathbf{y}_A \circ 1_X$$

(where 1_X is the identity on X). Thus $i^\#$ is an isomorphism with $\mathbf{y}_A$ as inverse.

3. $\Rightarrow$ 1. As we have already seen in the proof of Theorem 5.4, all elements of $\mathbf{y}\,(A)$ are finite in $\bar{A}$. Since $i^\#$ is isometric and surjective, all elements in $(i^\# \circ \mathbf{y})\,(A)$ are finite in X. Because $i = i^\# \circ \mathbf{y}$, all elements of A are finite in X. Since $i^\#$ is surjective, every element of X is the limit of a Cauchy sequence in A. Hence, A is a basis for X. □

A subset A of a generalized ultrametric space X for which the function $\mathbf{y}_A$ of the second clause above is isometric, is called *adequate* in [Law73] (p. 154).

This section is concluded by the introduction of the notion of adjoint pairs of mappings between gum's, and a characterization of completeness in terms thereof. This will not be used in the rest of the paper.

Let X and Y be generalized ultrametric spaces. A pair of non-expansive mappings $f : X \to Y$ and $g : Y \to X$ is *adjoint*, denoted by $f \dashv g$, if

$$\forall x \in X\ \forall y \in Y,\ \ Y(f(x), y) = X(x, g(y)).$$

An equivalent condition is that $X^X(1_X, g \circ f) = 0$ and $Y^Y(f \circ g, 1_Y) = 0$. Expressed in terms of the underlying orderings, this can be read as $1_X \le g \circ f$ and $f \circ g \le 1_Y$, saying that f and g are adjoint as monotone maps between the underlying preorders $\langle X, \le_X \rangle$ and $\langle Y, \le_Y \rangle$. For instance consider A^∞ with distance as defined in Section 2. Let $\Delta : A^\infty \to (A^\infty \times A^\infty)$ map v in A^∞ to $\langle v, v \rangle$, and let $\wedge : (A^\infty \times A^\infty) \to A^\infty$ map $\langle v, w \rangle$ to the longest common prefix of the words v and w. Then Δ is left adjoint to $\wedge$: for all v, w, and u in A^∞,

$$\max\{A^\infty(u, v),\ A^\infty(u, w)\} = A^\infty(u, v \wedge w).$$

(This defines a—$[0, 1]$-enriched—product on A^∞.)

The following lemma was suggested to us by Bart Jacobs.

Lemma 5.7 *Let X be a quasi ultrametric space. Consider the (corestriction of the) Yoneda embedding $\mathbf{y} : X \to \bar{X}$. The space X is complete if and only if there exists a non-expansive and continuous mapping $f : \bar{X} \to X$ with $f \dashv \mathbf{y}$.*

Proof: Suppose X is complete. By Theorem 5.5, there exists a unique non-expansive and continuous extension $1^\# : \bar{X} \to X$ of the identity mapping on X, defined, for $\phi = \lim \mathbf{y}(x_n)$ in $\bar{X}$ with $(x_n)_n$ a Cauchy sequence in X, by

$$1^\#(\phi) = \lim x_n.$$

For any $x \in X$,

$$\begin{array}{rcl} X(1^\#(\phi), x) & = & X(\lim x_n, x) \\ & = & \underleftarrow{\lim}\, X(x_n, x) \\ & = & \underleftarrow{\lim}\, \bar{X}(\mathbf{y}(x_n), \mathbf{y}(x)) \quad [\text{the Yoneda embedding is isometric}] \\ & = & \bar{X}(\lim \mathbf{y}(x_n), \mathbf{y}(x)) \\ & = & \bar{X}(\phi, \mathbf{y}(x)), \end{array}$$

showing that $1^\# \dashv \mathbf{y}$. For the converse suppose we are given a non-expansive and continuous mapping $f : \bar{X} \to X$ with $f \dashv \mathbf{y}$. For any Cauchy sequence $(x_n)_n$ in X and $x \in X$,

$$\begin{aligned} X(f(\lim \mathbf{y}(x_n)), x) &= \bar{X}(\lim \mathbf{y}(x_n), \mathbf{y}(x)) \\ &= \lim_{\leftarrow} \bar{X}(\mathbf{y}(x_n), \mathbf{y}(x)) \\ &= \lim_{\leftarrow} X(x_n, x) \quad \text{[the Yoneda embedding is isometric]}, \end{aligned}$$

proving that $\lim x_n = f(\lim \mathbf{y}(x_n))$. □

6 Topology via Yoneda

The Yoneda embedding of a generalized ultrametric space X into $\hat{X}$ gives rise to two topological closure operators. Their corresponding topologies are shown to generalize both the ϵ-ball topology of ordinary metric spaces and the Alexandroff and Scott topologies of preordered spaces.

Let X be a generalized ultrametric space. Recall that $\hat{X}$ is a generalized ultrametric space with the supremum distance, and that it contains as a subset an isometric copy of X via the Yoneda embedding. The fact that the Yoneda embedding is isometric justifies the following convention: we shall sometimes simply write x for $\mathbf{y}(x)$.

The main idea (stemming from [Law86]) is to interpret an element ϕ of $\hat{X}$ as a 'fuzzy' predicate (or 'fuzzy' subset) on X: the value that ϕ assigns to an element x in X is thought of as a measure for 'the extent to which x is an element of ϕ'. The smaller this number is, the more x should be viewed as an element. In fact, the only real elements are the ones where ϕ is 0, which gives rise to the definition of the *extension* $\int_A \phi$ of a predicate ϕ (the subscript A stands for 'Alexandroff' and will be explained below):

$$\int_A \phi = \{x \in X \mid \phi(x) = 0\}.$$

For instance, for x in X, $\int_A \mathbf{y}(x) = \int_A X(-, x) = \{z \in X \mid X(z, x) = 0\} = x{\downarrow}$. More generally, for any ϕ in $\hat{X}$,

$$\begin{aligned} \int_A \phi &= \{x \in X \mid \phi(x) = 0\} \\ &= \{x \in X \mid \hat{X}(X(-, x),\ \phi) = 0\} \quad \text{[the Yoneda Lemma 4.1]} \\ &= \{x \in X \mid \hat{X}(x,\ \phi) = 0\} \quad \text{[our convention]} \\ &= \phi{\downarrow} \cap X, \end{aligned}$$

where $\phi{\downarrow}$ is the downset of ϕ in $\hat{X}$ with respect to the underlying ordering. Any subset $V \subseteq X$ defines, conversely, a predicate $\rho_A(V) : X^{op} \to [0, 1]$ which is referred to as the *character* of the subset V. It is defined, for $x \in X$ by

$$\rho_A(V)(x) = \inf\{X(x, v) \mid v \in V\},$$

i.e., the distance from x to the set V. Note that under the identification of elements x in X with $X(-, x)$, this is equivalent to

$$\rho_A(V) = \inf V.$$

These two constructions define mappings $\int_A : \hat{X} \to \mathcal{P}(X)$ and $\rho_A : \mathcal{P}(X) \to \hat{X}$, which can be nicely related by considering $\hat{X}$ with the underlying preorder $\leq_{\hat{X}}$, and $\mathcal{P}(X)$ ordered by subset inclusion (cf. [Law86]):

Proposition 6.1 *The mappings $\int_A : \langle \hat{X}, \leq_{\hat{X}} \rangle \to \langle \mathcal{P}(X), \subseteq \rangle$ and $\rho_A : \langle \mathcal{P}(X), \subseteq \rangle \to \langle \hat{X}, \leq_{\hat{X}} \rangle$ are monotone. Moreover ρ_A is left adjoint to $\int_A$.*

Proof: Monotonicity of $\int_A$ and ρ_A follows directly from their definitions. We will hence concentrate on the second part of the proposition by proving for all $V \in \mathcal{P}(X)$ and $\phi \in \hat{X}$,

$$V \subseteq \textstyle\int_A \rho_A(V) \quad \text{and} \quad \rho_A(\textstyle\int_A \phi) \leq_{\hat{X}} \phi,$$

which is equivalent to ρ_A being left adjoint to $\int_A$, cf. [GHK$^+$80]. For $V \in \mathcal{P}(X)$ we have

$$\textstyle\int_A \rho_A(V) = \inf V{\downarrow} \cap X \supseteq V,$$

because $\inf V \geq_{\hat{X}} \mathbf{y}(x)$ for every $x \in V$. For $\phi \in \hat{X}$,

$$\rho_A(\textstyle\int_A \phi) = \inf(\phi{\downarrow} \cap X) = \inf\{\psi \in \hat{X} \mid \psi \leq_{\hat{X}} \phi\} \leq_{\hat{X}} \phi.$$

□

The above fundamental adjunction relates character of subsets and extension of predicates and is often referred to as the *comprehension schema* (cf. [Law73, Ken87]). As with any adjoint pair between preorders (cf. Theorem 0.3.6 of [GHK$^+$80]), the composition $\int_A \circ \rho_A$ is a closure operator on X. It satisfies, for $V \subseteq X$,

$$\begin{aligned}
\textstyle\int_A \circ \rho_A(V) &= (\inf V){\downarrow} \cap X \\
&= \{x \in X \mid \hat{X}(\mathbf{y}(x), \inf V) = 0\} \\
&= \{x \in X \mid \forall z \in X,\ \ [0,1](\mathbf{y}(x)(z),\ (\inf V)(z)) = 0\} \\
&= \{x \in X \mid \forall z \in X,\ \ \mathbf{y}(x)(z) \geq (\inf V)(z)\} \\
&= \{x \in X \mid \forall \epsilon > 0\ \forall z \in X,\ \ \mathbf{y}(x)(z) < \epsilon \Rightarrow (\exists v \in V,\ X(z,v) < \epsilon)\} \\
&= \{x \in X \mid \forall \epsilon > 0\ \forall z \in X,\ \ X(z,x) < \epsilon \Rightarrow (\exists v \in V,\ X(z,v) < \epsilon)\} \qquad (11) \\
& \text{[the Yoneda Lemma 4.1].}
\end{aligned}$$

As a consequence, there is the following lemma.

Lemma 6.2 *For a generalized ultrametric space X, the closure operator $\int_A \circ \rho_A : \mathcal{P}(X) \to \mathcal{P}(X)$ is a topological closure operator.*

Proof: It is an immediate consequence of (11) that $\int_A \circ \rho_A(\emptyset) = \emptyset$. Moreover, for $V, W \subseteq X$,

$$\textstyle\int_A \circ \rho_A(V \cup W) \quad \supseteq \quad \int_A \circ \rho_A(V) \cup \int_A \circ \rho_A(W),$$

because $\int_A \circ \rho_A$ is a closure operator. For the reverse inclusion, let $x \in \int_A \circ \rho_A(V \cup W)$. Suppose $x \notin \int_A \circ \rho_A(V)$. We will show $x \in \int_A \circ \rho_A(W)$: consider $\epsilon_1 > 0$ and $z_1 \in X$ with $X(z_1, x) < \epsilon_1$. We should find $y \in W$ with $X(z_1, y) < \epsilon_1$. Because $x \notin \int_A \circ \rho_A(V)$ there exist $\epsilon_0 > 0$ and $z_0 \in X$ such that

$$X(z_0, x) < \epsilon_0 \quad \& \quad (\forall y \in V,\ X(z_0, y) \geq \epsilon_0). \qquad (12)$$

Let $\epsilon = \min\{\epsilon_0, \epsilon_1\}$. Because $x \in \int \circ \rho(V \cup W)$ and $X(x,x) = 0 < \epsilon$, there exists $y \in V \cup W$ with $X(x,y) < \epsilon$. The assumption that $y \in V$ contradicts (12), because

$$X(z_0, y) \leq \max\{X(z_0, x), X(x, y)\} < \max\{\epsilon_0, \epsilon\} = \epsilon_0.$$

Thus $y \in W$. Furthermore,

$$X(z_1, y) \leq \max\{X(z_1, x), X(x, y)\} < \max\{\epsilon_1, \epsilon\} = \epsilon_1.$$

□

The above lemma implies that the closure operator $\int_A \circ \rho_A$ induces a topology on X, which in Proposition 6.3 below is proved equivalent to the following generalized ϵ-ball topology: For $x \in X$ and $\epsilon > 0$ define the *ϵ-ball centered in x* by

$$B_\epsilon(x) = \{z \in X \mid X(x, z) < \epsilon\}.$$

A subset $o \subseteq X$ of a generalized ultrametric space X is *generalized Alexandroff* open (gA-open) if for all $x \in X$

$$x \in o \quad \Rightarrow \exists \epsilon > 0\; B_\epsilon(x) \subseteq o.$$

The set of all gA-open subsets of X is denoted by $\mathcal{O}_{gA}(X)$. For instance, for every $x \in X$ the ϵ-ball $B_\epsilon(x)$ is a generalized Alexandroff open set. The pair $\langle X, \mathcal{O}_{gA}(X)\rangle$ can be shown to be topological space with $B_\epsilon(x)$, for every $\epsilon > 0$ and $x \in X$, as basic open sets (cf. [FK95]). For a subset V of X we write $cl_A(V)$ for the closure of V in the generalized Alexandroff topology.

Proposition 6.3 *For every subset V of a gum X, $cl_A(V) = \int_A \circ \rho_A(V)$.*

Proof: It follows from the characterization (11) of $\int_A \circ \rho_A$ that it is sufficient to prove

$$cl_A(V) = \{x \in X \mid \forall \epsilon > 0\, \forall z \in X,\;\; X(z,x) < \epsilon \;\Rightarrow\; (\exists v \in V,\; X(z,v) < \epsilon)\}.$$

Because $cl_A(V) = V \cup V^d$, where V^d is the so-called derived set of V (cf. Section A of the appendix), it follows from the definition of derived set and the fact that the set of all ϵ-balls is a basis for the generalized Alexandroff topology, that for every $x \in X$,

$$\begin{aligned} x \in V^d \quad &\iff \quad \forall o \in \mathcal{O}_{gA}(X),\;\; x \in o \;\Rightarrow\; o \cap (V \setminus \{x\}) \neq \emptyset \\ &\iff \quad \forall \epsilon > 0\, \forall z \in X,\;\; x \in B_\epsilon(z) \;\Rightarrow\; B_\epsilon(z) \cap (V \setminus \{x\}) \neq \emptyset \\ &\iff \quad \forall \epsilon > 0\, \forall z \in X,\;\; X(z,x) < \epsilon \;\Rightarrow\; \exists v \in (V \setminus \{x\}),\; X(z,v) < \epsilon. \end{aligned}$$

Therefore,

$$cl_A(V) = V \cup V^d = \{x \in X \mid \forall \epsilon > 0\, \forall z \in X,\;\; X(z,x) < \epsilon \;\Rightarrow\; (\exists v \in V,\; X(z,v) < \epsilon)\}.$$

$\square$

For ordinary ultrametric spaces, gA-open sets are just the usual open sets. For preorders, a set is gA-open precisely when it is Alexandroff open (upper closed) because if X is a preorder then for $\epsilon \leq 1$,

$$\begin{aligned} B_\epsilon(x) &= \{y \in X \mid X(x,y) < \epsilon\} \\ &= \{y \in X \mid X(x,y) = 0\} \\ &= \{y \in X \mid x \leq_X y\} \\ &= x\uparrow. \end{aligned}$$

(In case $\epsilon > 1$ then $B_\epsilon(x) = X$ which is clearly upper closed).

For computational reasons we are interested in complete spaces, in which one can model infinite behaviors by means of limits. A topology for a complete space X can then be considered satisfactory if limits in X are topological limits. This is not the case for the generalized Alexandroff topology: for instance, for complete partial orders $\mathcal{O}_{gA}(X)$ coincides with the standard Alexandroff topology, for which the coincidence of the least upperbounds of chains and their topological limits does not hold. Therefore the *Scott* topology is usually considered to be preferable: it is the coarsest topology refining the Alexandroff topology, in which least upper bounds of chains are topological limits (cf. Section II-1 of [GHK+80] and [Smy92]). Also for generalized ultrametric spaces, a suitable refinement of the generalized Alexandroff topology exists. A key step towards its definition is to compare fuzzy subsets ϕ in $\hat{X}$ with subsets of $\bar{X}$, the completion of X, rather than with subsets of X. To this end, the extension and the character functions of above are extended as follows:

$$\begin{array}{lcl} \int : \hat{X} \to \mathcal{P}(\bar{X}) & \text{and} & \rho : \mathcal{P}(\bar{X}) \to \hat{X}, \\ \quad \phi \mapsto \phi\!\downarrow \cap \bar{X} & & \qquad V \mapsto \inf V. \end{array}$$

Again we have a comprehension schema: as in Proposition 6.1, the mappings $\int : \langle \hat{X}, \leq_{\hat{X}}\rangle \to \langle \mathcal{P}(\bar{X}), \subseteq\rangle$ and $\rho : \langle \mathcal{P}(\bar{X}), \subseteq\rangle \to \langle \hat{X}, \leq_{\hat{X}}\rangle$ are monotone and ρ is left adjoint to $\int$. And again we obtain a closure operator, this time of type

$$\int \circ \rho : \mathcal{P}(\bar{X}) \to \mathcal{P}(\bar{X}),$$

which can, in a way similar to (11), be characterized as follows: for $V \subseteq \bar{X}$,

$$\begin{aligned}
\int \circ \rho(V) &= (\inf V){\downarrow} \cap \bar{X} \\
&= \{\phi \in \bar{X} \mid \hat{X}(\phi, \inf V) = 0\} \\
&= \{\phi \in \bar{X} \mid \forall a \in X,\ \ [0,1](\phi(a),\ (\inf V)(a)) = 0\} \\
&= \{\phi \in \bar{X} \mid \forall a \in X,\ \ \phi(a) \geq (\inf V)(a)\} \\
&= \{\phi \in \bar{X} \mid \forall \epsilon > 0\ \forall a \in X,\ \ \phi(a) < \epsilon \Rightarrow (\exists \psi \in V,\ \psi(a) < \epsilon)\} \\
&= \{\phi \in \bar{X} \mid \forall \epsilon > 0\ \forall a \in X,\ \ \bar{X}(a,\phi) < \epsilon \Rightarrow (\exists \psi \in V,\ \bar{X}(a,\psi) < \epsilon)\} \\
&\ [\text{the Yoneda Lemma 4.1}].
\end{aligned}$$

Also this closure operator is topological:

Lemma 6.4 *For a generalized ultrametric space X, the closure operator $\int \circ \rho : \mathcal{P}(\bar{X}) \to \mathcal{P}(\bar{X})$ is topological.*

Proof: This lemma is proved along the same lines as Lemma 6.2, but one needs the following additional observation: For any z_0 and z_1 in X, ϵ_0, $\epsilon_1 > 0$, and ϕ in $\bar{X}$, such that

$$\bar{X}(z_0, \phi) < \epsilon_0 \text{ and } \bar{X}(z_1, \phi) < \epsilon_1,$$

there exists $b \in X$ such that

$$\bar{X}(z_0, b) < \epsilon_0,\ \ \bar{X}(z_1, b) < \epsilon_1,\ \text{ and } \bar{X}(b, \phi) < \min\{\epsilon_0, \epsilon_1\}$$

(ϕ is now playing the role of x). This fact can be proved as follows. Because $\bar{X}$ is an algebraic complete quasi ultrametric space with (the image of the yoneda embedding of) X as basis, there exists a Cauchy sequence $(b_n)_n$ in X with $\phi = \lim b_n$. Since $z_0 \in X$, it is finite in $\bar{X}$. Hence, $\bar{X}(z_0, \phi) = \bar{X}(z_0, \lim(b_n)) < \epsilon_0$ implies the existence of N_0 such that for all $n \geq N_0$, $\bar{X}(z_0, b_n) < \epsilon_0$. Similarly, there exists N_1 such that for all $n \geq N_1$, $\bar{X}(z_1, b_n) < \epsilon_1$. Furthermore, there exists, by definition of limit, N_2 such that for all $n \geq N_2$, $\bar{X}(b_n, \phi) < \min\{\epsilon_0, \epsilon_1\}$. By taking $M = \max\{N_0, N_1, N_2\}$, and putting $b = b_M$, we have found the element in X we were looking for. □

Thus the closure operator above induces a topology on $\bar{X}$ which we will call the *generalized Scott* topology. Indeed, if we start out with an algebraic complete quasi ultrametric space X, then X is isomorphic to the completion of its basis B_X (by Proposition 5.6), and therefore the above characterization of $\int \circ \rho$ will take the form, for subsets $V \subseteq X$,

$$\int \circ \rho(V) = \{\phi \in X \mid \forall \epsilon > 0\ \forall a \in B_X,\ \ X(a, \phi) < \epsilon \Rightarrow (\exists \psi \in V,\ X(a, \psi) < \epsilon)\}. \tag{13}$$

In the special case that X is an algebraic complete partial order, this is equivalent to

$$\int \circ \rho(V) = \{x \in X \mid \forall a \in B_X,\ \ a \leq_X x \Rightarrow (\exists v \in V,\ a \leq_X v)\},$$

which we recognize as the closure operator induced by the Scott topology on X.

Next an alternative definition of the generalized Scott topology is given by specifying the open sets (this time starting with a complete generalized ultrametric space X). In Theorem 6.8 below, it will be shown that the closure operator induced by this second definition coincides with $\int \circ \rho$ whenever X is algebraic.

A subset $o \subseteq X$ of a complete generalized ultrametric space X is *generalized Scott* open (gS-open) if for all Cauchy sequences $(x_n)_n$ in X,

$$\lim x_n \in o \Rightarrow \exists N\ \exists \epsilon > 0\ \forall n \geq N,\ \ B_\epsilon(x_n) \subseteq o.$$

The set of all gS-open subsets of X is denoted by $\mathcal{O}_{gS}(X)$. Below it will be shown that this defines a topology indeed. Note that every gS-open set $o \subseteq X$ is gA-open because every point $x \in X$ is the limit of the constant Cauchy sequence $(x)_n$ in X. Therefore this topology refines the generalized Alexandroff topology. Furthermore it will be shown to

1. coincide with the ϵ-ball topology in case X is a complete ultrametric space; and to

2. coincide with the Scott topology in case X is a complete partial order.

The following proposition gives an example of gS-open sets:

Proposition 6.5 *For every complete generalized ultrametric space X, an element $a \in X$ is finite if and only if for every $\epsilon > 0$, the set $B_\epsilon(a)$ is gS-open.*

Proof: Let $a \in X$ be finite and $\epsilon > 0$. Then the ϵ-ball $B_\epsilon(a)$ is a gS-open set: let $(x_n)_n$ be a Cauchy sequence in X and assume $\lim x_n \in B_\epsilon(a)$. Because a is finite, $X(a, \lim x_n) = \lim X(a, x_n) < \epsilon$, by which there exists $\delta' > 0$ such that $\lim X(a, x_n) < \epsilon - \delta'$. Take $\delta < \delta'$. Then there exists $N \geq 0$ such that $X(a, x_n) < (\epsilon - \delta') + \delta$, for all $n \geq N$. Then $B_\delta(x_n) \subseteq B_\epsilon(a)$ for all $n \geq N$, because if $y \in B_\delta(x_n)$ for some $n \geq N$ we have, by triangular inequality and by our choice of δ,

$$X(a, y) \leq \max\{X(a, x_n), X(x_n, y)\} < \max\{((\epsilon - \delta') + \delta), \delta\} = (\epsilon - \delta') + \delta < \epsilon,$$

that is, $y \in B_\epsilon(a)$.

Conversely, assume $B_\epsilon(a)$ is a gS-open set for every $\epsilon > 0$. We need to prove, for every Cauchy sequence $(x_n)_n$ in X, that

$$\lim X(a, x_n) \leq X(a, \lim x_n) \tag{14}$$

(the reverse inequality is given by Proposition 3.5). If $\lim X(a, x_n) = 0$ then (14) is trivially true. Therefore suppose $\lim X(a, x_n) > 0$ and, towards a contradiction, assume $X(a, \lim x_n) < \lim X(a, x_n)$. Then there exists $\epsilon > 0$ such that $X(a, \lim x_n) < \epsilon < \lim X(a, x_n)$. Moreover

$$\begin{array}{rcll} X(a, \lim x_n) < \epsilon & \Rightarrow & \lim x_n \in B_\epsilon(a) & \\ & \Rightarrow & \exists N\, \exists \delta > 0\, \forall n \geq N,\ B_\delta(x_n) \subseteq B_\epsilon(a) & [B_\epsilon(a) \text{ is a gS-open set}] \\ & \Rightarrow & \exists N\, \forall n \geq N, X(a, x_n) < \epsilon & \\ & \Longleftrightarrow & \lim X(a, x_n) < \epsilon. & \end{array}$$

But this contradicts $\epsilon < \lim X(a, x_n)$. Thus $X(a, \lim x_n) \geq \lim X(a, x_n)$. □

Next we prove that the collection of all gS-open sets forms indeed a topology.

Proposition 6.6 *For every complete generalized ultrametric space X the pair $\langle X, \mathcal{O}_{gS}(X)\rangle$ is a topological space. If X is also algebraic with basis B_X, then the set $\{B_\epsilon(a) \mid a \in B_X \ \&\ \epsilon > 0\}$ forms a basis for the generalized Scott topology $\mathcal{O}_{gS}(X)$.*

Proof: We first prove that $\mathcal{O}_{gS}(X)$ is closed under finite intersections and arbitrary unions. Let I be a finite index set (possibly empty) and let $o = \bigcap_I o_i$ with $o_i \in \mathcal{O}_{gS}(X)$ for all $i \in I$. If $\lim x_n \in o$ for a Cauchy sequences $(x_n)_n$ in X, then for every $i \in I$ there exist $N_i \geq 0$ and $\epsilon_i > 0$ such that $B_{\epsilon_i}(x_n) \subseteq o_i$ for all $n \geq N_i$. Take $N = \max_I N_i$ and $\epsilon = \min_I \epsilon_i$ (here $\max_\emptyset = 0$ and $\min_\emptyset = 1$). Then $B_\epsilon(x_n) \subseteq o$ for all $n \geq N$, that is, o is gS-open.

Next let I be an arbitrary index set and let $o = \bigcup_I o_i$ with $o_i \in \mathcal{O}_{gS}(X)$ for all $i \in I$. If $\lim x_n \in o$ for a Cauchy sequences $(x_n)_n$ in X, then there exists $i \in I$ such that $\lim x_n \in o_i$. Therefore there exists $N \geq 0$ and $\epsilon > 0$ such that $B_\epsilon(x_n) \subseteq o_i \subseteq o$ for all $n \geq N$, that is, o is gS-open.

Finally assume that X is an algebraic complete gum with basis B_X. We have already seen that for every $\epsilon > 0$ and finite element $a \in B_X$ the set $B_\epsilon(a)$ is gS-open. We claim that every gS-open set $o \subseteq X$ is the union of ϵ-balls of finite elements. Let $x \in o$. Since X is algebraic there is a Cauchy sequence $(a_n)_n$ in B_X with $x = \lim a_n$. Because o is gS-open, there exists $\epsilon_x > 0$ and $N_x \geq 0$ such that $B_{\epsilon_x}(a_n) \subseteq o$ for all $n \geq N_x$ and with $x \in B_{\epsilon_x}(a_n)$ for N_x big enough. Therefore $o \subseteq \bigcup_{x \in o} B_{\epsilon_x}(a_{N_x})$. Since the other inclusion trivially holds we have that the collection of all ϵ-ball of finite elements forms a basis for the generalized Scott topology. □

Any ordinary complete ultrametric space X is an algebraic complete generalized ultrametric space where all elements are finite. Therefore, by the previous proposition, the basic open sets of the generalized Scott topology are all the ϵ-balls $B_\epsilon(x)$, with $x \in X$. Hence for ordinary complete ultrametric spaces the generalized Scott topology coincides with the standard ϵ-ball topology.

For a complete partial order X, a set $o \subseteq X$ is gS-open precisely when it is Scott open: if $o \in \mathcal{O}_{gS}(X)$ then it is upper closed because the gS-topology refines the gA-topology. Moreover, if $\bigsqcup x_n \in o$ for an ω-chain $(x_n)_n$ in X then—because o is gS-open—there exists $\epsilon > 0$ and $N \geq 0$ such that $B_\epsilon(x_n) \subseteq o$ for all $n \geq N$. But $x_n \in B_\epsilon(x_n)$ for all ϵ, therefore o is an ordinary Scott open set. Conversely, assume o is Scott open and let $\lim x_n \in o$. Because o is Scott open (and limits are least upper bounds) there exists $N \geq 0$ such that $x_n \in o$ for all $n \geq N$. By taking $\epsilon = 1/2$ we obtain that o is also gS-open because for every $x \in X$, $B_{1/2}(x) = x\uparrow$.

The specialization preorder on an algebraic complete generalized ultrametric space X induced by its gS-topology coincides with the preorder underlying X:

Proposition 6.7 *For an algebraic complete generalized ultrametric space X and x and y in X,*

$$x \leq_{\mathcal{O}_{gS}} y \;\Leftrightarrow\; x \leq_X y \;\; (\Leftrightarrow_{def} \; X(x,y) = 0).$$

Proof: Let $x \leq_{\mathcal{O}_{gS}} y$. Since X is algebraic there exists a Cauchy sequence $(b_n)_n$ of finite elements such that $x = \lim b_n$. By definition of limit, for every $\epsilon > 0$ there exists $N \geq 0$ such that for all $n \geq N$, $X(b_n, x) < \epsilon$, that is $x \in B_\epsilon(b_n)$. But $B_\epsilon(b_n)$ is gS-open because b_n is finite, thus also $y \in B_\epsilon(b_n)$ since $x \leq_{\mathcal{O}_{gS}} y$. Therefore for every $\epsilon > 0$ there exists $N \geq 0$ such that for all $n \geq N$, $X(b_n, y) < \epsilon$, from which it follows that

$$X(x,y) = X(\lim b_n, y) = \lim_{\leftarrow} X(b_n, y) = 0.$$

Conversely, assume $X(x,y) = 0$ and let o be a gS-open set such that $x \in o$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq o$. But $y \in B_\epsilon(x)$ for every ϵ because $X(x,y) = 0$, therefore $y \in o$. $\square$

Note that the specialization preorder is a *partial* order—or, equivalently the gS-topology is $\mathcal{T}_0$—if and only if X is an algebraic complete quasi ultrametric space.

As usual, a subset c of a complete gum X is *gS-closed* if its complement $X \setminus c$ is gS-open. This is equivalent to the following condition: for all Cauchy sequences $(x_n)_n$ in X,

$$(\forall N \, \forall \epsilon > 0 \, \exists n \geq N \, \exists y \in c, \;\; X(x_n, y) < \epsilon) \;\Rightarrow\; \lim x_n \in c. \tag{15}$$

For a subset V of X we write $cl_S(V)$ for the closure of V in the generalized Scott topology, that is, $cl_S(V)$ is the smallest generalized Scott closed set containing V. From the definition of limits we have that for any Cauchy sequence $(x_n)_n$ in V, $\lim x_n \in cl_S(V)$. The latter implies that if X is a generalized ultrametric space with basis B then B is *dense* in X, that is $cl_S(B) = X$. Indeed, $B \subseteq X$ implies $cl_S(B) \subseteq cl_S(X) = X$. For the converse we use the fact that every element of X is the limit of a Cauchy sequence in B. Since (the image under $\mathbf{y}$ of) every generalized ultrametric space X is a basis for its completion $\bar{X}$ it follows that every gum is dense in its completion.

As promised above, it is shown that cl_S and $\int \circ \rho$ are equal.

Theorem 6.8 *Let X be an algebraic complete quasi ultrametric space X with basis B_X. For all subsets $V \subseteq X$, $cl_S(V) = \int \circ \rho(V)$.*

Proof: It follows from the characterization (13) of $\int \circ \rho$ that it is sufficient to prove that

$$cl_S(V) = \{x \in X \mid \forall \epsilon > 0 \, \forall a \in B_X, \;\; X(a,x) < \epsilon \;\Rightarrow\; (\exists v \in V, \; X(a,v) < \epsilon)\}.$$

We use the fact that $cl_S(V) = V \cup V^d$ for $V \subseteq X$, where V^d is the derived set with respect to the gS-topology. Since the ϵ-balls of finite elements form a basis for the generalized Scott topology, we have for every $x \in X$:

$$\begin{aligned} x \in V^d \quad &\Longleftrightarrow \quad \forall o \in \mathcal{O}_{gS}(X), \;\; x \in o \;\Rightarrow\; o \cap (V \setminus \{x\}) \neq \emptyset \\ &\Longleftrightarrow \quad \forall \epsilon > 0 \, \forall a \in B_X, \;\; x \in B_\epsilon(a) \;\Rightarrow\; B_\epsilon(a) \cap (V \setminus \{x\}) \neq \emptyset \\ &\Longleftrightarrow \quad \forall \epsilon > 0 \, \forall a \in B_X, \;\; X(a,x) < \epsilon \;\Rightarrow\; \exists v \in (V \setminus \{x\}), \; X(a,v) < \epsilon. \end{aligned}$$

Therefore,

$$cl_S(V) = V \cup V^d = \{x \in X \mid \forall \epsilon > 0\ \forall a \in B_X,\ \ X(a,x) < \epsilon\ \Rightarrow\ (\exists v \in V,\ X(a,v) < \epsilon)\}.$$

□

This section is concluded with two observations relating metric limits and topological convergence (cf. Section A). We start by showing that in a complete generalized ultrametric space every Cauchy sequence is topologically convergent to its limit. It is an open problem if the converse holds—without completeness it does not hold even for standard metric spaces (cf. [Dug66, Example XIV.I.1, pag 293]). First note that it is straightforward from the definition of convergence that a sequence $(x_n)_n$ in an algebraic complete generalized ultrametric space X converges (with respect to the gS-topology on X) to an element x in X, denoted by $\mathcal{N}((x_n)_n) \to x$, if and only if

$$\forall \epsilon > 0\ \forall a \in B_X,\ \ X(a,x) < \epsilon\ \Rightarrow\ \ (\exists N\ \forall n \geq N,\ X(a,x_n) < \epsilon).$$

Proposition 6.9 *Let X be a complete generalized ultrametric space and $(x_n)_n$ a Cauchy sequence in X. Then $\mathcal{N}((x_n)_n) \to \lim x_n$. If X is moreover algebraic then for any $y \in X$ such that $\mathcal{N}((x_n)_n) \to y$ it holds $y \leq_{\mathcal{O}_{gS}} \lim x_n$, that is, limits are maximal topological limits.*

Proof: Let $(x_n)_n$ be a Cauchy sequence in the complete generalized ultrametric space X. We need to prove $\mathcal{N}(\lim x_n) \subseteq \mathcal{N}((x_n)_n)$. By definition of gS-open set,

$$o \in \mathcal{N}(\lim x_n) \iff \lim x_n \in o \Rightarrow \exists N\ \exists \epsilon > 0\ \forall n \geq N,\ B_\epsilon(x_n) \subseteq o,$$

from which $o \in \mathcal{N}((x_n)_n)$ follows immediately.

Next assume that X is also algebraic with basis B_X and let $\mathcal{N}((x_n)_n) \to y$ for a Cauchy sequence $(x_n)_n$ in X and $y \in X$. Since X is algebraic there exists a Cauchy sequence $(a_m)_m$ in B_X with $y = \lim a_m$. Therefore for every $\epsilon > 0$ there exists $M \geq 0$ such that $X(a_m, y) < \epsilon$ for all $m \geq M$. Hence $y \in B_\epsilon(a_m)$ which is a gS-open set and hence in $\mathcal{N}(y) \subseteq \mathcal{N}((x_n)_n)$. Thus

$$\forall \epsilon > 0\ \exists M \geq 0\ \forall m \geq M \exists N \geq 0\ \forall n \geq N,\ X(a_m, x_n) < \epsilon,$$

which implies $\forall \epsilon > 0\ \exists M \geq 0\ \forall m \geq M,\ \lim_n X(a_m, x_n) < \epsilon$. Since all the a_m's are finite we then have $\forall \epsilon > 0\ \exists M \geq 0\ \forall m \geq M,\ X(a_m, \lim x_n) < \epsilon$, which means $\lim X(a_m, \lim x_n) = 0$. Finally, by the definition of limit,

$$0 = \underset{\leftarrow}{\lim} X(a_m, \lim x_n) = X(\lim a_m, \lim x_n) = X(y, \lim x_n),$$

which implies, by Proposition 6.7, $y \leq_{\mathcal{O}_{gS}} \lim x_n$. □

Recall that a function $f : X \to Y$ between two complete generalized ultrametric spaces is (metrically) continuous if $f(\lim x_n) = \lim f(x_n)$ for every Cauchy sequence $(x_n)_n$ in X. It is *topologically* continuous if the inverse image of a gS-open subset of Y is gS-open in X. The two notions are related as follows.

Proposition 6.10 *Let $f : X \to Y$ be a non-expansive mapping between complete generalized ultrametric spaces. If f is metrically continuous then it is also topologically continuous. Moreover, if Y is an algebraic complete quasi ultrametric then the converse holds as well.*

Proof: Let $f : X \to Y$ be a non-expansive and metrically continuous function and let $o \in \mathcal{O}_{gS}(Y)$. We need to prove $f^{-1}(o) \in \mathcal{O}_{gS}(X)$ in order to conclude that f is topologically continuous. Indeed, for any Cauchy sequence $(x_n)_n$ in X we have

$$\begin{array}{rcl}
\lim x_n \in f^{-1}(o) & \iff & f(\lim x_n) \in o \\
& \iff & \lim f(x_n) \in o \quad [f \text{ is metrically continuous}] \\
& \Rightarrow & \exists N\ \exists \epsilon > 0\ \forall n \geq N,\ B_\epsilon(f(x_n)) \subseteq o \\
& & [f \text{ is non-expansive, } (f(x_n))_n \text{ is a Cauchy sequence, } o \text{ is gS-open}] \\
& \Rightarrow & \exists N\ \exists \epsilon > 0\ \forall n \geq N,\ B_\epsilon(x_n) \subseteq f^{-1}(o) \quad [f \text{ is non-expansive}].
\end{array}$$

For the converse assume Y to be an algebraic complete quasi ultrametric space and $f : X \to Y$ to be topologically continuous. Any Cauchy sequence $(x_n)_n$ in X converges, by Proposition 6.9 to $\lim x_n$. Hence $(f(x_n))_n$ converges to $f(\lim x_n)$ because f is topologically continuous. But, by Proposition 6.9, $(f(x_n))_n$ converges also to $\lim f(x_n)$ and since Y is algebraic then $f(\lim x_n) \leq_{\mathcal{O}_{gS}} \lim f(x_n)$. Therefore by Proposition 6.7, $Y(f(\lim x_n), \lim f(x_n)) = 0$. Also $Y(\lim f(x_n), f(\lim x_n)) = 0$ because f is non-expansive:

$$\begin{aligned} Y(\lim f(x_n), f(\lim x_n)) &= \underleftarrow{\lim}\, Y(f(x_n), f(\lim x_n)) \\ &\leq \underleftarrow{\lim}\, X(x_n, \lim x_n) \\ &= X(\lim x_n, \lim x_n) \\ &= 0. \end{aligned}$$

Since Y is a quasi ultrametric and $Y(f(\lim x_n), \lim f(x_n)) = 0 = Y(\lim f(x_n), f(\lim x_n))$ we finally obtain $\lim f(x_n) = f(\lim x_n)$. $\square$

7 Powerdomains via Yoneda

A generalized lower (or *Hoare*) powerdomain for generalized ultrametric spaces is defined, again by means of the Yoneda embedding. Next this powerdomain is characterized in terms of completion and topology. Also the definition of generalized upper and convex powerdomains will be given. Their characterizations will be discussed elsewhere.

Let $\oplus : [0,1] \times [0,1] \to [0,1]$ map elements r and s in $[0,1]$ to (their coproduct) $\min\{r, s\}$. This makes $\langle [0,1], \oplus \rangle$ a *semi-lattice*: for all r, s, and t in $[0,1]$,

$$(i)\ r \oplus r = r, \quad (ii)\ r \oplus s = s \oplus r, \quad (iii)\ (r \oplus s) \oplus t = r \oplus (s \oplus t).$$

Furthermore, the following inequality holds for all r and s in $[0,1]$:

$$(iv)\ r \leq_{[0,1]} r \oplus s.$$

Let X be a generalized ultrametric space. It is immediate that $\langle \hat{X}, \oplus \rangle$ is a semi-lattice as well, with $\oplus$ taken pointwise: for ϕ and ψ in $\hat{X}$ and x in X,

$$(\phi \oplus \psi)(x) = \psi(x) \oplus \psi(x).$$

Recalling the idea that elements in $\hat{X}$ are fuzzy subsets of X, the semi-lattice operation $\oplus$ may be viewed as fuzzy subset union. A *generalized lower powerdomain* on $\bar{X}$ is now defined as the smallest subset of $\hat{X}$ which contains the image of X under the Yoneda embedding $\mathbf{y} : X \to \hat{X}$ (Corollary 4.2); is metrically complete (i.e., contains limits of Cauchy sequences); and is closed under the operation $\oplus$. Formally,

$$\mathcal{P}_{gl}(\bar{X}) = \bigcap \{V \subseteq \hat{X} \mid \mathbf{y}(X) \subseteq V,\ V \text{ is a complete subspace of } \hat{X}, \text{ and } V \text{ is closed under } \oplus\}.$$

This definition is very similar to the definition of completion in Section 5.

Note that the above definition of a powerdomain applies to arbitrary algebraic complete quasi ultrametric spaces Y, since for any basis A for Y, $Y \cong \bar{A}$.

A generalized Hausdorff distance

The powerdomain $\mathcal{P}_{gl}(\bar{X})$ can be described in a number of ways. The main tool will be the adjunction of Section 6:

$$\textstyle\int : \hat{X} \to \mathcal{P}(\bar{X}), \quad \phi \mapsto \phi{\downarrow} \cap \bar{X}; \text{ and } \rho : \mathcal{P}(\bar{X}) \to \hat{X}, \quad V \mapsto \inf V,$$

which relates $\hat{X}$ to the collection of subsets of $\bar{X}$, the completion of X. Before turning to the characterizations of $\mathcal{P}_{gl}(\bar{X})$, let us first show how this adjunction induces a distance on $\mathcal{P}(\bar{X})$: for subsets V and W of $\bar{X}$, define

$$\mathcal{P}(\bar{X})(V, W) = \hat{X}(\rho(V), \rho(W)).$$

It satisfies the following equation.

Theorem 7.1 *For all V and W in $\mathcal{P}(\bar{X})$,*

$$\mathcal{P}(\bar{X})(V, W) = \inf\{\epsilon > 0 \mid \forall a \in X\ \forall v \in V,\ \hat{X}(a, v) < \epsilon \Rightarrow (\exists w \in W,\ \hat{X}(a, w) < \epsilon)\}.$$

Proof: Let I denote the set on the right of the equality. In order to show that $\mathcal{P}(\bar{X})(V, W) \leq \inf I$ consider $\epsilon \in I$. If $V = \emptyset$ then $\mathcal{P}(\bar{X})(V, W) = 0 \leq \inf I$. Next let $v = \lim a_n$ be an element of V, with a_n in X, for all n. Let δ a real number with $0 < \delta < \epsilon$. It follows from $v = \lim a_n$ that

$$\hat{X}(v, \inf W) = \underleftarrow{\lim}\, \hat{X}(a_n, \inf W).$$

Therefore there exists $n > 0$ such that

$$\hat{X}(a_n, v) < \delta \text{ and } \hat{X}(v, \inf W) \leq \hat{X}(a_n, \inf W) + \delta.$$

Because $\delta < \epsilon$, it follows from $\hat{X}(a_n, v) < \delta$ that there exists $w \in W$ with $\hat{X}(a_n, w) < \epsilon$. Therefore,

$$\begin{aligned}
\hat{X}(v, \inf W) &\leq \hat{X}(a_n, \inf W) + \delta \\
&\leq \inf W(a_n) + \delta \quad \text{[Yoneda lemma]} \\
&= \inf_{u \in W} u(a_n) + \delta \\
&= \inf_{u \in W} \hat{X}(a_n, u) + \delta \quad \text{[Yoneda lemma]} \\
&\leq \hat{X}(a_n, w) + \delta \\
&\leq \epsilon + \delta.
\end{aligned}$$

Since δ was arbitrary this implies $\hat{X}(v, \inf W) \leq \epsilon$, from which it follows that

$$\begin{aligned}
\mathcal{P}(\bar{X})(V, W) &= \hat{X}(\inf V, \inf W) \\
&= \sup_{v \in V} \hat{X}(v, \inf W) \quad \text{[see Lemma 7.2 below]} \\
&\leq \epsilon.
\end{aligned}$$

Hence $\mathcal{P}(\bar{X})(V, W) \leq \inf I$.

For the reverse let $\delta > 0$ be arbitrary and define

$$\epsilon = \mathcal{P}(\bar{X})(V, W) + \delta.$$

We shall show that $\epsilon \in I$. Because δ is arbitrary this will imply that $\inf I \leq \mathcal{P}(\bar{X})(V, W)$. Consider $a \in X$ and $v \in V$ with $\hat{X}(a, v) < \epsilon$. We have to prove that there exists $w \in W$ such that $\hat{X}(a, w) < \epsilon$:

$$\begin{aligned}
\hat{X}(a, \inf W) &\leq \max\{\hat{X}(a, v),\ \hat{X}(v, \inf W)\} \\
&\leq \max\{\hat{X}(a, v),\ \sup_{u \in V} \hat{X}(u, \inf W)\} \\
&= \max\{\hat{X}(a, v),\ \hat{X}(\inf V, \inf W)\} \quad \text{[see Lemma 7.2 below]} \\
&= \max\{\hat{X}(a, v),\ \mathcal{P}(\bar{X})(V, W)\} \\
&< \epsilon.
\end{aligned}$$

As before, the Yoneda lemma implies $\hat{X}(a, \inf W) = \inf_{w \in W} \hat{X}(a, w)$. Thus there is w in W such that $\hat{X}(a, w) < \epsilon$. □

The following lemma, used above, is an immediate consequence of Lemma 3.3.

Lemma 7.2 *For any $V \subseteq \bar{X}$ and $\phi \in \hat{X}$, $\hat{X}(\inf V, \phi) = \sup_{v\in V} \hat{X}(v, \phi)$.*

Proof: For $V \subseteq \bar{X}$ and $\phi \in \hat{X}$,

$$\begin{aligned}
&\hat{X}(\inf V, \phi) \\
&= \sup_{x\in X} [0,1]((\inf V)(x),\ \phi(x)) \\
&= \sup_{x\in X} [0,1](\inf_{v\in V} v(x),\ \phi(x)) \\
&= \sup_{x\in X} \sup_{v\in V} [0,1](v(x),\ \phi(x)) \quad \text{[Lemma 3.3]} \\
&= \sup_{v\in V} \sup_{x\in X} [0,1](v(x),\ \phi(x)) \\
&= \sup_{v\in V} \hat{X}(v, \phi).
\end{aligned}$$

□

Another equivalent description of $\mathcal{P}(\bar{X})(V, W)$ can be obtained as a corollary of Theorem 7.1:

$$\begin{aligned}
\mathcal{P}(\bar{X})(V, W) &= \hat{X}(\inf V, \inf W) \\
&= \sup_{v\in V} \hat{X}(v, \inf W) \quad \text{[Lemma 7.2]} \\
&= \sup_{v\in V} \hat{X}(\inf\{v\}, \inf W) \\
&= \sup_{v\in V} \mathcal{P}(\bar{X})(\{v\}, W) \\
&= \sup_{v\in V} \inf\{\epsilon > 0 \mid \forall a \in X,\ \hat{X}(a, v) < \epsilon \Rightarrow (\exists w \in W, \hat{X}(a, w) < \epsilon)\} \\
&\quad \text{[Theorem 7.1]}.
\end{aligned}$$

Therefore the above distance on $\mathcal{P}(\bar{X})$ is called the *generalized Hausdorff* distance. The restriction of the distance on $\mathcal{P}(\bar{X})$ to subsets of X gives the familiar (non-symmetric) Hausdorff distance (cf. [Law86]). More precisely:

Theorem 7.3 *For all $V \subseteq \bar{X}$ and $W \subseteq \bar{X}$ such that either $V \subseteq X$ or W is finite,*

$$\mathcal{P}(\bar{X})(V, W) = \sup_{v\in V} \inf_{w\in W} \bar{X}(v, w).$$

Proof: Applying the Yoneda lemma twice gives, for all v in X, $\inf_{w\in W} \hat{X}(v, w) = \hat{X}(v, \inf W)$. If W is finite the same equality holds for arbitrary $v \in \bar{X}$ (by an extension of Lemma 7.2 similar to Lemma 3.3). Therefore, if either $V \subseteq X$ or W is finite,

$$\begin{aligned}
\sup_{v\in V} \inf_{w\in W} \hat{X}(v, w) &= \sup_{v\in V} \hat{X}(v, \inf W) \\
&= \hat{X}(\inf V, \inf W) \quad \text{[Lemma 7.2]} \\
&= \mathcal{P}(\bar{X})(V, W).
\end{aligned}$$

□

For preorders X, the above amounts to

$$V \leq_{\mathcal{P}(\bar{X})} W \text{ iff } \forall v \in V\ \exists w \in W,\ v \leq_X w,$$

which is the usual Hoare ordering. More generally, for a gum X, there is the following characterization of the order induced by $\mathcal{P}(\bar{X})$.

Lemma 7.4 *For every gum X and subsets V and W of $\bar{X}$: if W is gS-closed then*

$$V \leq_{\mathcal{P}(\bar{X})} W \textit{ if and only if } V \subseteq W \ .$$

Proof: If $V \subseteq W$ then $\mathcal{P}(\bar{X})(V, W) = 0$ by Theorem 7.1. Conversely, assume $\mathcal{P}(\bar{X})(V, W) = 0$ and let $x \in V$. Because $x \in \bar{X}$, there exists a Cauchy sequence $(b_n)_n$ in X with $x = \lim \mathbf{y}(b_n)$. Thus for every $\epsilon > 0$ there exists $N > 0$ such that $\bar{X}(\mathbf{y}(b_n), x) < \epsilon$ for all $n \geq N$. By, again, Theorem 7.1, and the fact that $\mathcal{P}(\bar{X})(V, W) = 0$, we then obtain

$$\forall \epsilon > 0 \exists N > 0 \forall n \geq N \exists w \in W, \bar{X}(\mathbf{y}(b_n), w) < \epsilon,$$

which implies

$$\forall \epsilon > 0 \forall N > 0 \exists n \geq N \exists w \in W, \bar{X}(\mathbf{y}(b_n), w) < \epsilon.$$

By the characterization of closed sets given by formula (15) in Section 6, it follows that $x = \lim \mathbf{y}(b_n) \in W$. □

Because $V \subseteq cl_S(V)$, for every $V \subseteq \bar{X}$, the above lemma implies $\mathcal{P}(\bar{X})(V, cl_S(V)) = 0$. Also $\mathcal{P}(\bar{X})(cl_S(V), V) = 0$: this follows from Theorem 7.1 and the characterization of the generalized Scott closure operator, which states that for every $x \in cl_S(V)$, every $\epsilon > 0$ and $b \in X$, if $\hat{X}(b, x) < \epsilon$ then there exists $v \in V$ with $\hat{X}(b, v) < \epsilon$. This leads to the following.

Lemma 7.5 *For every gum X, and subsets V, W of $\bar{X}$,*

$$\mathcal{P}(\bar{X})(V, W) = \mathcal{P}(\bar{X})(cl_S(V), W) \textit{ and } \mathcal{P}(\bar{X})(V, W) = \mathcal{P}(\bar{X})(V, cl_S(W)).$$

Proof: Immediate from the fact that $\mathcal{P}(\bar{X})(V, cl_S(V)) = 0 = \mathcal{P}(\bar{X})(cl_S(V), V)$, and the triangular inequality. □

Characterizing $\mathcal{P}_{gl}(\bar{X})$ as a completion

Let $\mathcal{P}_{nf}(X)$ be the generalized ultrametric space consisting of all finite and nonempty subsets of X with the non-symmetric Hausdorff distance defined above: for V and W in $\mathcal{P}_{nf}(X)$,

$$\begin{aligned} \mathcal{P}_{nf}(X)(V, W) &= \hat{X}(\rho(V), \rho(W)) \\ &= \max_{v \in V} \min_{w \in W} X(v, w) \quad \text{[by Theorem 7.3]}. \end{aligned}$$

Its completion $\overline{\mathcal{P}_{nf}(X)}$ will be shown to be isomorphic to $\mathcal{P}_{gl}(\bar{X})$.

Lemma 7.6 *For any generalized ultrametric space X,*

$$\overline{\mathcal{P}_{nf}(X)} \cong \{\lim \rho(V_n) \mid V_n \in \mathcal{P}_{nf}(X), \textit{ for all } n, \textit{ and } (\rho(V_n))_n \textit{ is Cauchy in } \hat{X}\}.$$

Proof: Let us denote the set on the right by R. Because the quasi ultrametric space $\widehat{X}$ is complete, the isometric, and hence non-expansive, function $\rho : \mathcal{P}_{nf}(X) \to \widehat{X}$ induces a non-expansive and continuous function $\rho^{\#} : \overline{\mathcal{P}_{nf}(X)} \to \widehat{X}$ according to Theorem 5.5, making the following diagram commute:

$$\begin{tikzcd} \mathcal{P}_{nf}(X) \arrow[r, "\mathbf{y}"] \arrow[dr, "\rho"'] & \overline{\mathcal{P}_{nf}(X)} \arrow[d, dotted, "\rho^{\#}"] \\ & \widehat{X} \end{tikzcd}$$

It follows from Proposition 5.2 that the image of $\rho^{\#}$ is precisely R. Furthermore $\rho^{\#}$ is isometric: for all Cauchy sequences $(V_n)_n$ and $(W_m)_m$ in $\mathcal{P}_{nf}(X)$,

$$\begin{aligned}
&\widehat{X}\left(\rho^{\#}\left(\lim_n \mathbf{y}\left(V_n\right)\right), \rho^{\#}\left(\lim_m \mathbf{y}\left(W_m\right)\right)\right) \\
&\quad = \widehat{X}\left(\lim_n \rho\left(V_n\right), \lim_m \rho\left(W_m\right)\right) \\
&\quad = \underleftarrow{\lim}_n \lim_m \widehat{X}\left(\rho\left(V_n\right), \rho\left(W_m\right)\right) \quad [\rho\left(V_n\right) \text{ is finite in } \widehat{X} \text{ by Lemma 7.7 below}] \\
&\quad = \underleftarrow{\lim}_n \lim_m \mathcal{P}_{nf}(X)\left(V_n, W_m\right) \quad [\rho \text{ is isometric}] \\
&\quad = \underleftarrow{\lim}_n \lim_m \widehat{\mathcal{P}_{nf}(X)}\left(\mathbf{y}\left(V_n\right), \mathbf{y}\left(W_m\right)\right) \quad [\mathbf{y} \text{ is isometric}] \\
&\quad = \widehat{\mathcal{P}_{nf}(X)}\left(\lim_n \mathbf{y}\left(V_n\right), \lim_m \mathbf{y}\left(W_m\right)\right) \quad [\mathbf{y}\left(V_n\right) \text{ is finite in } \widehat{\mathcal{P}_{nf}(X)}] \\
&\quad = \overline{\mathcal{P}_{nf}(X)}\left(\lim_n \mathbf{y}\left(V_n\right), \lim_m \mathbf{y}\left(W_m\right)\right).
\end{aligned}$$

Thus $\rho^{\#}$ is injective and hence an isomorphism from $\overline{\mathcal{P}_{nf}(X)}$ to R. □

The following lemma, used in the proof above, generalizes Lemma 4.3.

Lemma 7.7 *For any V in $\mathcal{P}_{nf}(X)$, $\rho(V)$ is finite in $\widehat{X}$.*

Proof We only treat the case that $V = \{v_1, v_2\}$ (the general case follows by induction on the number of elements of V). For any Cauchy sequence $(\phi_n)_n$ in $\widehat{X}$,

$$\begin{aligned}
&\widehat{X}(\rho(V), \lim \phi_n) \\
&\quad = \widehat{X}(\min\{\mathbf{y}(v_1), \mathbf{y}(v_2)\}, \lim \phi_n) \\
&\quad = \max\{\widehat{X}(\mathbf{y}(v_1), \lim \phi_n),\ \widehat{X}(\mathbf{y}(v_2), \lim \phi_n)\} \quad [\text{Lemma 7.2}] \\
&\quad = \max\{\lim \widehat{X}(\mathbf{y}(v_1), \phi_n),\ \lim \widehat{X}(\mathbf{y}(v_2), \phi_n)\} \quad [\text{Lemma 4.3}] \\
&\quad = \lim\ \max\{\widehat{X}(\mathbf{y}(v_1), \phi_n),\ \widehat{X}(\mathbf{y}(v_2), \phi_n)\} \\
&\quad = \lim \widehat{X}(\min\{\mathbf{y}(v_1), \mathbf{y}(v_2)\}, \phi_n) \\
&\quad = \lim \widehat{X}(\rho(V), \phi_n).
\end{aligned}$$

□

The following theorem will be often used in the sequel.

Theorem 7.8 *For any generalized ultrametric space X,*

$$\mathcal{P}_{gl}(\bar{X}) = \{\lim \rho(V_n) \mid V_n \in \mathcal{P}_{nf}(X),\ \textit{for all } n, \textit{ and } (\rho(V_n))_n \textit{ is Cauchy in } \hat{X}\}.$$

Proof: Let R again denote the righthand side. Because R contains $\mathbf{y}(X)$, is complete (by Lemma 7.6), and is closed under $\oplus$:

$$\begin{aligned}
\lim \rho(V_n) \oplus \lim \rho(W_n) &= \lim(\rho(V_n) \oplus \rho(W_n)) \quad [\oplus \text{ is continuous on } \hat{X}] \\
&= \lim \rho(V_n \cup W_n),
\end{aligned}$$

for Cauchy sequences $(\rho(V_n))_n$ and $(\rho(W_n))_n$, it follows that $\mathcal{P}_{gl}(\bar{X}) \subseteq R$.

For the converse note that any subset V of $\hat{X}$ which is closed under $\oplus$ and contains $\mathbf{y}(X)$, also contains $\rho(V)$ for any $V \in \mathcal{P}_{nf}(X)$. If V is moreover complete than $\lim \rho(V_n)$ is in V, for any Cauchy sequence $(\rho(V_n))_n$ in $\hat{X}$ with $V_n \in \mathcal{P}_{nf}(X)$, for all n. Consequently, R is contained in any V having all three properties. Thus $R \subseteq \mathcal{P}_{gl}(\bar{X})$. □

Combining Lemma 7.6 and Theorem 7.8 yields the following.

Corollary 7.9 *For any generalized ultrametric space X, $\mathcal{P}_{gl}(\bar{X}) \cong \overline{\mathcal{P}_{nf}(X)}$.* □

The above description of the generalized lower powerdomain can be used to give the following categorical characterization. Let a *metric lower semi-lattice* be an algebraic complete quasi ultrametric space X together with a non-expansive and continuous operation $+ : X \times X \to X$ such that, for all x, y, and z in X,

$$(i)\ x + x = x, \quad (ii)\ x + y = y + x, \quad (iii)\ (x + y) + z = x + (y + z), \quad (iv)\ x \leq_X x + y\ .$$

For example, $\langle \mathcal{P}_{gl}(\bar{X}), \oplus \rangle$ is a metric lower semi-lattice because $\mathcal{P}_{gl}(\bar{X})$ is an algebraic complete quasi ultrametric space by the above corollary, and $\oplus$ is continuous and non-expansive.

As a consequence of Theorem 7.8, the lower powerdomain construction can be seen to be free. First note that every x in a gum X is mapped by $\mathbf{y} : X \to \hat{X}$ to an element of $\mathcal{P}_{gl}(\bar{X})$. Thus we may consider $\mathbf{y}$ as a non-expansive map $\mathbf{y} : X \to \mathcal{P}_{gl}(\bar{X})$. Since $\mathcal{P}_{gl}(\bar{X})$ is an algebraic complete quasi ultrametric space, Theorem 5.5 gives us a non-expansive and continuous function $\mathbf{y}^{\#} : \bar{X} \to \mathcal{P}_{gl}(\bar{X})$. It is used in the following.

Theorem 7.10 *For any gum X, metric lower semi-lattice $\langle Y, + \rangle$, and non-expansive and continuous function $f : \bar{X} \to Y$ there exists a unique non-expansive, continuous and + preserving mapping $f^\star : \langle \mathcal{P}_{gl}(\bar{X}), \oplus \rangle \to \langle Y, + \rangle$ such that $f^\star \circ \mathbf{y}^{\#} = f$:*

$$\begin{tikzcd} \bar{X} \arrow[r, "\mathbf{y}^{\#}"] \arrow[dr, "f"'] & \mathcal{P}_{gl}(\bar{X}) \arrow[d, dotted, "f^\star"] \\ & Y. \end{tikzcd}$$

□

(This theorem can be proved similarly to Theorem 5.5.)

Now let $Lsl(Acq)$ denote the category of metric lower semi-lattices with continuous, non-expansive and + preserving functions as morphisms. There is a forgetful functor $\mathcal{U} : Lsl(Acq) \to Acq$ which maps every metric lower semi-lattices $\langle Y, + \rangle$ to Y. As a consequence of Theorem 7.10, the lower powerdomain construction can be extended to a functor $\mathcal{P}_{gl}(-) : Acq \to Lsl(Acq)$ which is left adjoint to $\mathcal{U}$. As usual, this implies that the functor $\mathcal{U} \circ \mathcal{P}_{gl}(-) : Acq \to Acq$ is locally non-expansive and locally continuous (cf. [Plo83, Rut95]), by which it can be used in the construction of recursive domain equations.

Characterizing $\mathcal{P}_{gl}(\bar{X})$ topologically

The main result of this subsection is that for generalized ultrametric spaces X *that are countable*:

$$\mathcal{P}_{gl}(\bar{X}) \cong \mathcal{P}^+_{gS}(\bar{X}),$$

where

$$\mathcal{P}^+_{gS}(\bar{X}) = \{V \subseteq \bar{X} \mid V \text{ is gS-closed and non-empty }\}.$$

The proof makes use of the adjunction $\rho \vdash \int$ as follows. As with any adjunction between preorders, the co-restrictions of ρ and $\int$ give an isomorphism

$$\rho : Im(\textstyle\int) \to Im(\rho), \quad \textstyle\int : Im(\rho) \to Im(\textstyle\int).$$

Recall that the gS-closed subsets of $\bar{X}$ are precisely the fixed points of $\int \circ \rho$ (Theorem 6.8). Because $\int \circ \rho \circ \int = \int$ (as with any adjunction between preorders), all elements of $Im(\int)$ are gS-closed. Thus

$$\begin{aligned} \mathcal{P}_{gS}(\bar{X}) &= \{V \subseteq \bar{X} \mid V \text{ is gS-closed }\} \\ &= \{V \subseteq \bar{X} \mid V = \textstyle\int \circ \rho(V)\} \\ &= Im(\textstyle\int). \end{aligned}$$

In order to conclude that $\mathcal{P}_{gl}(\bar{X}) \cong \mathcal{P}^+_{gS}(\bar{X})$, it is now sufficient to prove $\mathcal{P}_{gl}(\bar{X}) = Im^+(\rho)$, where

$$Im^{+}(\rho) = \{\rho(V) \in \hat{X} \mid V \subseteq \bar{X},\ V \text{ nonempty }\}.$$

This will be a consequence of the following lemma and theorem.

The inclusion $\mathcal{P}_{gl}(\bar{X}) \subseteq Im^{+}(\rho)$ is an immediate consequence of Theorem 7.8 and the following.

Lemma 7.11 *For all Cauchy sequences $(\rho(V_n))_n$ in $\hat{X}$ such that V_n is a finite and nonempty subset of X for all n, $\lim \rho(V_n) \in Im^{+}(\rho)$.*

Proof: Let $(V_n)_n$ be a sequence of finite and nonempty subsets of X such that $(\rho(V_n))_n$ is Cauchy in $\hat{X}$. We shall prove: $\lim \rho(V_n) = \rho(\{\lim v_n \mid v_n \in V_n,\ \text{for all } n,\ \text{and } (v_n)_n \text{ is Cauchy in } X\})$. (It will follow from the proof below that the set on the right is nonempty.) Let $(v_n)_n$, with $v_n \in V_n$ be a Cauchy sequence in X. For all n, $\rho(V_n) \leq v_n$ (in $\hat{X}$ taken with the pointwise extension of the standard ordering on $[0,1]$). Therefore $\lim \rho(V_n) \leq \lim v_n$. Because $(v_n)_n$ is arbitrary, this implies

$$\lim \rho(V_n) \leq \rho(\{\lim v_n \mid v_n \in V_n,\ \text{for all } n,\ \text{and } (v_n)_n \text{ is Cauchy in } X\}).$$

For the converse let $x \in X$ and $\epsilon > 0$. We shall construct a Cauchy sequence $(v_n)_n$ in X such that

$$\lim v_n(x) \leq \lim \rho(V_n)(x) + 2 \cdot \epsilon.$$

Let N be such that for all $n \geq N$,

$$\hat{X}(\rho(V_N), \rho(V_n)) \leq \epsilon, \text{ and } \rho(V_N)(x) \leq \lim \rho(V_n)(x) + \epsilon.$$

Choose v_i in V_i arbitrarily, for $0 \leq i < N$. Because V_N is finite there exists $v_N \in V_N$ such that $\rho(V_N)(x) = X(x, v_N) = v_N(x)$. Choose v_{N+1} in V_{N+1} such that

$$X(v_N, v_{N+1}) = \min_{w \in V_{N+1}} X(v_N, w).$$

Because, by Theorem 7.3,

$$\hat{X}(\rho(V_N), \rho(V_{N+1})) = \max_{v \in V_N} \min_{w \in V_{N+1}} X(v, w),$$

it follows that

$$X(v_N, v_{N+1}) \leq \hat{X}(\rho(V_N), \rho(V_{N+1})) \leq \epsilon.$$

Continuing this way, we find a sequence $(v_n)_n$ in X which is Cauchy because $(\rho(V_n))_n$ is. Now for all $n \geq N$, $[0,1](v_N(x), v_n(x)) \leq \epsilon$, or equivalently, $v_n(x) \leq \max\{\epsilon, v_N(x)\}$. Thus

$$\begin{aligned}
\lim v_n(x) &\leq \max\{\epsilon, v_N(x)\} \\
&\leq v_N(x) + \epsilon \\
&= \rho(V_N)(x) + \epsilon \\
&\leq \lim \rho(V_n)(x) + 2 \cdot \epsilon.
\end{aligned}$$

□

The reverse inclusion: $Im^{+}(\rho) \subseteq \mathcal{P}_{gl}(\bar{X})$, is a consequence of Theorem 7.8 and the following.

Theorem 7.12 *Let X be countable. For any subset nonempty V of $\bar{X}$ there exists a sequence $(V_n)_n$ of finite and nonempty subsets of X such that $\rho(V) = \lim\ \rho(V_n)$ in $\hat{X}$.*

Proof: Let $V \subseteq \bar{X}$, nonempty. We shall define a sequence $(V_n)_n$ of finite and (eventually) nonempty subsets of X such that for any $\phi \in \hat{X}$,

$$\hat{X}(\rho(V), \phi) = \underset{\leftarrow}{\lim}\, \hat{X}(\rho(V_n), \phi).$$

The proof proceeds in five steps as follows.

1. Let $x_1, x_2, \ldots$ be an enumeration of X. The sets V_n are defined by induction on n. They will consist of elements of X which are approximations of elements of V. More precisely, they will satisfy, for all $n \geq 1$,

$$\forall x \in V_n,\ B_{1/n}(x) \cap V \neq \emptyset.$$

For convenience, we start at $n = 1$. Let

$$V_1 = \begin{cases} \{x_1\} & \text{if } B_1(x_1) \cap V \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Now suppose we have already defined V_n. We assume: for all $x \in V_n$, $B_{1/n}(x) \cap V \neq \emptyset$. In the construction of V_{n+1}, we shall include for every element of the previously constructed set V_n again an element (possibly the same), which will be a better approximation of the set V. Moreover, we shall take into account x_{n+1}, the $(n+1)$-th element in the enumeration of X. Let

$$V_{n+1} = \{\text{improve}(x) \mid x \in V_n\} \cup \{\text{represent}(x_{n+1}) \mid B_1(x_{n+1}) \cap V \neq \emptyset\},$$

where 'improve(x)' and 'represent(x_{n+1})' are defined as follows:

- If $B_{1/n+1}(x) \cap V \neq \emptyset$ then put improve$(x) = x$: x is still 'good enough'. Otherwise consider $y \in V$ with $\hat{X}(x, y) < 1/n$, which exists by the inductive hypothesis that $B_{1/n}(x) \cap V \neq \emptyset$. Let $y = \lim y_k$, with y_k in X for all k. Because x is in X it is finite in $\hat{X}$, whence

  $$\hat{X}(x, y) = \lim \hat{X}(x, y_k).$$

  Therefore we can choose a number k big enough such that

  $$\hat{X}(y_k, y) < 1/(n+1) \text{ and } \hat{X}(x, y_k) < 1/n.$$

  Define improve$(x) = y_k$. Note that

  $$B_{1/(n+1)}(\text{improve}(x)) \cap V \neq \emptyset \text{ and } \hat{X}(x, \text{improve}(x)) < 1/n.$$

- Suppose $B_1(x_{n+1}) \cap V \neq \emptyset$. (If this does not hold the second set in the definition of V_{n+1} is empty.) If $B_{1/(n+1)}(x_{n+1}) \cap V \neq \emptyset$ then x_{n+1} is close enough to V, and we put: represent$(x_{n+1}) = x_{n+1}$. Otherwise let i be the maximal natural number smaller than $n+1$ such that $B_{1/i}(x_{n+1}) \cap V \neq \emptyset$. Let $y \in V$ be such that $\hat{X}(x_{n+1}, y) < 1/i$. Let $y = \lim y_k$, with y_k in X for all k. As before we can choose a number k such that

  $$\hat{X}(y_k, y) < 1/(n+1) \text{ and } \hat{X}(x_{n+1}, y_k) < 1/i,$$

  and put: represent$(x_{n+1}) = y_k$. Note that

  $$B_{1/(n+1)}(\text{represent}(x_{n+1})) \cap V \neq \emptyset \text{ and } \hat{X}(x_{n+1}, \text{represent}(x_{n+1})) < 1/i.$$

For all $x \in V_{n+1}$, $B_{1/(n+1)}(x) \cap V \neq \emptyset$. Because V is nonempty there exists N such that for all $n \geq N$, V_n is nonempty.

2. Some properties of $(V_n)_n$: Because $\hat{X}(x, \text{improve}(x)) < 1/n$, for all $n \geq 1$ and $x \in V_n$, it follows that

$$\begin{aligned} \hat{X}(\rho(V_n), \rho(V_{n+1})) &= \sup_{v \in V_n} \inf_{w \in V_{n+1}} \hat{X}(v, w) \quad \text{[Theorem 7.3]} \\ &< 1/n. \end{aligned}$$

Because $B_{1/n}(x) \cap V \neq \emptyset$, for all $n \geq 1$ and $x \in V_n$, also

$$\hat{X}(\rho(V_n), \rho(V)) < 1/n.$$

3. As a consequence, $(\rho(V_n))_n$ is a Cauchy sequence in $\hat{X}$. Since for all $n \geq 1$ and $\phi \in \hat{X}$,

$$\begin{aligned}\hat{X}(\rho(V_n), \phi) &\leq \max\{\hat{X}(\rho(V_n), \rho(V)),\ \hat{X}(\rho(V), \phi)\} \\ &\leq \hat{X}(\rho(V), \phi) + 1/n,\end{aligned}$$

it follows that

$$\lim\limits_{\leftarrow} \hat{X}(\rho(V_n), \phi) \leq \hat{X}(\rho(V), \phi).$$

4. Next we shall prove the converse:

$$\hat{X}(\rho(V), \phi) \leq \lim\limits_{\leftarrow} \hat{X}(\rho(V_n), \phi).$$

Because $\hat{X}(\rho(V), \phi) = \hat{X}(\inf V, \phi) = \sup_{v \in V} \hat{X}(v, \phi)$ it will be sufficient to prove for all $y \in V$,

$$\hat{X}(y, \phi) \leq \lim\limits_{\leftarrow} \hat{X}(\rho(V_n), \phi).$$

So let $y = \lim y_m$ be in V with $y_m \in X$ for all m. Let $M \geq 1$ be arbitrary. Choose m big enough such that

$$\begin{aligned}\hat{X}(y, \phi) &= \hat{X}(\lim y_m, \phi) \\ &= \lim\limits_{\leftarrow} \hat{X}(y_m, \phi) \\ &\leq \hat{X}(y_m, \phi) + 1/M,\end{aligned}$$

and $\hat{X}(y_m, y) < 1/M$. Let $k \geq 1$ be such that $y_m = x_k$. (Recall that $X = \{x_1, x_2, \ldots\}$.) We distinguish between the following two cases:

(i) $k \leq M$: Because $1/M \leq 1/k$ it follows from the construction of $(V_n)_n$ that $x_k \in V_k,\ x_k \in V_{k+1}, \ldots, x_k \in V_M$. Therefore

$$\begin{aligned}\hat{X}(y_m, \phi) &= \hat{X}(x_k, \phi) \\ &\leq \sup_{x \in V_M} \hat{X}(x, \phi) \\ &= \hat{X}(\inf V_M, \phi) \\ &= \hat{X}(\rho(V_M), \phi).\end{aligned}$$

Because $\hat{X}(\rho(V_N), \lim \rho(V_n)) \leq 1/M$, for $N \geq M$ big enough,

$$\begin{aligned}\hat{X}(\rho(V_M), \phi) &\leq \max\{\hat{X}(\rho(V_M), \rho(V_N)),\ \hat{X}(\rho(V_N), \lim \rho(V_n)),\ \hat{X}(\lim \rho(V_n), \phi)\} \\ &\leq \hat{X}(\lim \rho(V_n), \phi) + 1/M \\ &= \lim\limits_{\leftarrow} \hat{X}(\rho(V_n), \phi) + 1/M,\end{aligned}$$

which implies

$$\hat{X}(y_m, \phi) \leq \lim\limits_{\leftarrow} \hat{X}(\rho(V_n), \phi) + 1/M.$$

(ii) $M < k$: If $B_{1/k}(x_k) \cap V = B_{1/k}(y_m) \cap V \neq \emptyset$ then represent$(x_k) = x_k$. Otherwise let i be the maximal number below k such that $B_{1/i}(x_k) \cap V \neq \emptyset$. Because $\hat{X}(x_k, y) = \hat{X}(y_m, y) < 1/M$ it follows that $M \leq i$, whence

$$\hat{X}(x_k, \mathrm{represent}(x_k)) < 1/i \leq 1/M.$$

Therefore we have, whether $B_{1/k}(x_k) \cap V$ is empty or nonempty,

$$\begin{array}{rcl}\hat{X}(y_m, \phi) & = & \hat{X}(x_k, \phi) \\ & \leq & \max\{\hat{X}(x_k, \mathrm{represent}(x_k))\ \hat{X}(\mathrm{represent}(x_k), \phi)\} \\ & \leq & \hat{X}(\mathrm{represent}(x_k), \phi) + 1/M \\ & \leq & \sup_{x \in V_k} \hat{X}(x, \phi) + 1/M \\ & = & \hat{X}(\rho(V_k), \phi) + 1/M.\end{array}$$

Because, as in case (i),

$$\hat{X}(\rho(V_k), \phi) \leq \lim_{\leftarrow} \hat{X}(\rho(V_n), \phi) + 1/k,$$

and $1/k < 1/M$, this implies

$$\hat{X}(y_m, \phi) \leq \lim_{\leftarrow} \hat{X}(\rho(V_n), \phi) + 2/M.$$

It follows that in both cases

$$\hat{X}(y, \phi) \leq \hat{X}(y_m, \phi) + 1/M \leq \lim_{\leftarrow} \hat{X}(\rho(V_n), \phi) + 3/M.$$

Because M is arbitrary, this implies

$$\hat{X}(y, \phi) \leq \lim_{\leftarrow} \hat{X}(\rho(V_n), \phi).$$

5. We have shown:

$$\hat{X}(\rho(V), \phi) = \lim_{\leftarrow} \hat{X}(\rho(V_n), \phi).$$

□

Lemma 7.11 and Theorem 7.12, together with Theorem 7.8, imply:

Corollary 7.13 *For a countable generalized ultrametric space X, $\mathcal{P}_{gl}(\bar{X}) = Im^+(\rho)$.* □

All in all, we have:

Theorem 7.14 *For a countable generalized ultrametric space X, $\mathcal{P}_{gl}(\bar{X}) \cong \mathcal{P}^+_{gS}(\bar{X})$.*

Proof: The isomorphism $\mathcal{P}_{gS}(\bar{X}) \cong Im(\rho)$ restricts to an isomorphism $\mathcal{P}^+_{gS}(\bar{X}) \cong Im^+(\rho)$. By Corollary 7.13, $\mathcal{P}_{gl}(\bar{X}) = Im^+(\rho)$. Therefore, $\mathcal{P}_{gl}(\bar{X}) \cong \mathcal{P}^+_{gS}(\bar{X})$. □

It follows that if we start with an ω-algebraic complete quasi ultrametric space X with basis B_X (for which $X \cong \overline{B_X}$), then

$$\mathcal{P}_{gl}(X) \cong \{V \subseteq X \mid V \text{ is gS-closed and nonempty }\}.$$

Using the characterization of $\mathcal{P}_{gl}(X)$ as a completion, it follows that $\mathcal{P}_{gl}(X)$ is an ω-algebraic complete quasi ultrametric space with as (countable) basis the set

$$\{cl_S(V) \mid V \in \mathcal{P}_{nf}(B_X)\}.$$

The collection of closed sets of a given topological space X often comes with the *lower topology* [Mic51, Nad78]. Recall that given a topological space $\langle X, \mathcal{O}(X)\rangle$, the lower topology $\mathcal{O}_L(S)$ on a collection of subset $S \subseteq \mathcal{P}(X)$ is defined by taking the collection of sets of the form

$$L_o = \{V \in S \mid V \cap o \neq \emptyset\},$$

for all $o \in \mathcal{O}(X)$, as a subbasis. This subsection is concluded by showing that for an ω-algebraic complete quasi ultrametric space X, the lower topology on $\mathcal{P}_{gS}(X)$ and the generalized Scott topology on $\mathcal{P}_{gS}(X)$ coincide.

Theorem 7.15 *For an ω-algebraic complete quasi ultrametric space X,*

$$\mathcal{O}_L(\mathcal{P}_{gS}(X)) = \mathcal{O}_{gS}(\mathcal{P}_{gS}(X)).$$

Proof: Let B_X be a countable basis for X. Let $o \in \mathcal{O}_{gS}(X)$ and consider the sub-basic open set $L_o \in \mathcal{O}_L(\mathcal{P}_{gS}(X))$. A gS-closed set V is in L_o if and only if $V \cap o \neq \emptyset$ or, equivalently, $V \not\subseteq X \setminus o$. Because $X \setminus o$ is gS-closed, it follows from Lemma 7.4 that $\mathcal{P}(X)(V, X \setminus o) \neq 0$. Therefore,

$$V \in L_o \iff V \in \{W \in \mathcal{P}_{gS}(X) \mid \mathcal{P}(X)(W, X \setminus o) \neq 0\}.$$

But the rightmost set is open in the gS-topology of $\mathcal{P}_{gS}(X)$ because it is the complement of the gS-closed set

$$cl_S(\{X \setminus o\}) = \{V \in \mathcal{P}_{gS}(X) \mid \mathcal{P}(X)(W, X \setminus o) = 0\}$$

(the latter equality being a consequence of Lemma 6.7 and Lemma 7.4). This proves $\mathcal{O}_L(\mathcal{P}_{gS}(X)) \subseteq \mathcal{O}_{gS}(\mathcal{P}_{gS}(X))$.

For the converse, let V be a finite subset of B_X and consider, for some $\epsilon > 0$, the basic open set $B_\epsilon(cl_S(V))$ of the gS-topology on $\mathcal{P}_{gS}(X)$. For any $W \in \mathcal{P}_{gS}(X)$,

$$\begin{aligned}
& W \in B_\epsilon(cl_S(V)) \\
\iff\quad & \mathcal{P}(X)(cl_S(V), W) < \epsilon \\
\iff\quad & \mathcal{P}(X)(V, W) < \epsilon \quad \text{[Lemma 7.5]} \\
\iff\quad & \sup_{b \in V} \inf_{x \in W} X(b, x) < \epsilon \quad \text{[Theorem 7.3, } V \subseteq B_X\text{]} \\
\iff\quad & \forall b \in V,\ \inf_{x \in W} X(b, x) < \epsilon. \\
\iff\quad & \forall b \in V,\ W \cap B_\epsilon(b) \neq \emptyset \\
\iff\quad & W \in \bigcap_{b \in V} L_{B_\epsilon(b)} \quad [B_\epsilon(b) \text{ is basic open in } \mathcal{O}_{gS}(X)].
\end{aligned}$$

Since V is finite, the above proves that every basic open set of $\mathcal{O}_{gS}(\mathcal{P}_{gS}(X))$ can be expressed as the intersection of finitely many sub-basic open sets of $\mathcal{O}_L(\mathcal{P}_{gS}(X))$. Thus $\mathcal{O}_{gS}(\mathcal{P}_{gS}(X)) \subseteq \mathcal{O}_L(\mathcal{P}_{gS}(X))$. □

Generalized upper and convex powerdomains

We briefly sketch the construction of a generalized upper and convex powerdomain. They will be treated in detail elsewhere.

Let X be a generalized ultrametric space. A *generalized upper powerdomain* on $\bar{X}$ can be defined dually to $\mathcal{P}_{gl}(\bar{X})$ as follows. First $[0,1]$ is considered again as a semi-lattice, now with $\otimes : [0,1] \times [0,1] \to [0,1]$ sending elements r and s in $[0,1]$ to (their product) $\max\{r, s\}$. Next let

$$\check{X} = ([0,1]^X)^{op}.$$

It can be turned into a semi-lattice $\langle \check{X}, \otimes \rangle$ by taking the pointwise extension of $\otimes$. There is the following dual version of the Yoneda embedding:

$$\check{\mathbf{y}} : X \to \check{X}, \quad x \mapsto X(x, -),$$

where $X(x, -)$ maps y in X to $X(x, y)$. Now the generalized upper powerdomain is given by

$$\mathcal{P}_{gu}(\bar{X}) = \bigcap \{V \subseteq \check{X} \mid \check{\mathbf{y}}(X) \subseteq V,\ V \text{ is a complete subspace of } \check{X}, \text{ and } V \text{ is closed under } \otimes\}.$$

Also this powerdomain can be characterized in a number of ways, one of which is via completion: Consider again $\mathcal{P}_{nf}(X)$, this time with distance, for all V and W in $\mathcal{P}_{nf}(X)$,

$$\mathcal{P}_{nf}(X)(V,W) = \sup_{w\in W} \inf_{v\in V} X(v,w).$$

Then the completion of $\mathcal{P}_{nf}(X)$ is isomorphic to $\mathcal{P}_{gu}(\bar{X})$. In the special case that X is a preorder, this amounts to the standard definition of the upper, or Smyth, powerdomain.

A *generalized convex powerdomain* is obtained by combining the constructions of the generalized lower and upper powerdomains (thus using both the Yoneda embedding and its dual). It can again be easily described as the completion of $\mathcal{P}_{nf}(X)$, now taken with distance

$$\mathcal{P}_{nf}(X)(V,W) = \max\{\sup_{v\in V} \inf_{w\in W} X(v,w),\ \sup_{w\in W} \inf_{v\in V} X(v,w).\}$$

For a preorder X, the convex powerdomain coincides with the standard convex, or Plotkin, powerdomain; for an ordinary ultrametric space, it yields the powerdomain of compact subsets.

8 Related work

The thesis that fundamental structures are categories has been the main motivation for Lawvere in his study of generalized metric spaces as enriched categories [Law73]. Lawvere's work together with the more topological perspective of Smyth [Smy87] have been our main source of inspiration for the present paper which continues the work of Rutten [Rut95]. Generalized ultrametric spaces are a special instance of Lawvere's $\mathcal{V}$-categories. The non-symmetric ultrametric for $[0,1]$ is also described and studied in his paper. The notion of forward Cauchy sequence for a non-symmetric metric space is from [Smy87] as well as the notion of limit. A purely enriched categorical definition of forward Cauchy sequences and of limits can be found in Wagner's [Wag94, Wag95]. The notion of finiteness and algebraicity for a generalized ultrametric space are from [Rut95].

Clearly we are working in the tradition of domain theory, for which Plotkin's [Plo83] has been our main source of information.

Completion and topology of non-symmetric metric spaces have been extensively studied in [Smy87], seeking to reconcile metric spaces and complete partial orders as topological spaces by considering quasi-uniformities. Smyth gives criteria for the appropriateness of a topology for a quasi-uniform space. Also a completion by means of Cauchy sequences is present in his work. The main difference with our work is the simplicity of the theory of generalized metric spaces obtained by the enriched categorical perspective, in particular by the use of the Yoneda Lemma. Indeed, both the categorical perspective of Lawvere and the topological one of Smyth have been combined in our approach to obtain a reconciliation of complete metric spaces with complete partial orders.

The fact that the Yoneda lemma gives rise to completion is well known for many mathematical structures such as groups, lattices, and categories. In [Wag95], an enriched version of the Dedekind-MacNeille completion of lattices is given. In [SMM95], the Yoneda lemma is used in the definition of a completion of monoidal closed categories. The use of the Yoneda lemma for the completion of generalized metric spaces is new, but it is suggested by an embedding theorem of Kuratowski [Kur35] and the definition of completion as in [Eng89, Theorems 4.3.13-4.3.19] for standard metric spaces. A metric version of the Yoneda lemma also occurs, though not under that name, in [JMP86, Lemma II-2.8]. The comprehension schema as a comparison between predicates and subsets has been studied in the context of generalized metric spaces by Lawvere [Law73] and Kent [Ken87]. The definition of the generalized Scott topology via the Yoneda embedding seems to be new while the direct definition—by specifying the open sets—is briefly mentioned in the conclusion of [Smy87]. A generalized Scott topology is also given in [Wag95]. However his notion of topology does not coincide with the standard one: for example it is not the ϵ-ball topology in the case of standard metric spaces.

Another important topological approach to quasi metric spaces which needs to be mentioned is that of, again, Smyth [Smy91] and Flagg and Kopperman [FK95]. They consider quasi metric spaces equipped with the generalized Alexandroff topology. In order to reconcile metric spaces with complete partial orders they assign to partial orders a distance function which, in general, is not discrete. Their approach to topology, completion and powerdomains is much simpler than ours because much of the standard metric topological theorems can be adapted. The price to be paid for such simplicity is that this approach only works for a restricted class of spaces: they have to be spectral. Hence a full reconciliation between metric spaces and partial orders is not possible (e.g., only algebraic cpo's which are so-called 2/3 SFP are spectral in their Scott topology). Also the work of Sünderhauf on quasi-uniformities [Sün94] is along the same lines.

The study of powerdomains for complete generalized metric spaces is new. Some results on the restricted class of totally bounded quasi metric spaces are due to Smyth's [Smy91] and Flagg and Kopperman's [FK95]. The lower powerdomain has also been studied by Kent [Ken87] but for generalized metric spaces which need not be complete. Our use of the Yoneda embedding for defining the powerdomains and for their topological characterization is new. It is inspired by the work of Lawvere [Law73, Law86].

Other papers on reconciling complete partial orders and metric spaces are [WS81, CD85, Mat94]. In [RSV82] seven distinct notions of Cauchy sequences can be found. For one of these notions of Cauchy sequence—but different from ours—completion has been studied in [Doi88].

Acknowledgements

The authors are grateful to the following persons for suggestions, comments, and discussions: Jaco de Bakker, Pietro di Gianantonio, Bart Jacobs, Bill Rounds, Erik de Vink, and Kim Wagner. Marcello Bonsangue and Jan Rutten have enjoyed the opportunity to present this work at an early stage at a seminar at LITP (Paris), organized by Paul Gastin and Maurice Nivat.

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A Topological preliminaries

A *topology* $\mathcal{O}(X)$ on a set X is a collection of subsets of X that is closed under finite intersections and arbitrary unions. The pair $\langle X, \mathcal{O}(X)\rangle$ is called a *topological space* and every $o \in \mathcal{O}(X)$ is called an *open set* of the space X. A set is *closed* if its complement is open. A *base* of a topology $\mathcal{O}(X)$ on X is a set $\mathcal{B} \subseteq \mathcal{O}(X)$ such that every open set is the union of elements of $\mathcal{B}$. A *subbase* of $\mathcal{O}(X)$ is a set $\mathcal{S} \subseteq \mathcal{O}(X)$ such that the collection of finite intersections of elements in $\mathcal{S}$ is a basis of $\mathcal{O}(X)$.

Every topology $\mathcal{O}(X)$ on a set X induces a preorder on X called the *specialization preorder*: for any x and y in X, $x \leq_{\mathcal{O}} y$ if and only if

$$\forall o \in \mathcal{O}(X), \;\; x \in o \Rightarrow y \in o.$$

A topology is called $\mathcal{T}_0$ if the specialization preorder is a partial order.

A *closure operator* on a set X is a function $cl : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ such that, for all S and S' in $\mathcal{P}(X)$,

(i) $S \subseteq cl(S)$ (ii) $cl(S) = cl(cl(S))$

(iii) if $S \subseteq S'$ then $cl(S) \subseteq cl(S')$

A closure operator is strict if $cl(\emptyset) = \emptyset$. A *topological closure operator* is a strict closure operator cl that moreover is finitely additive: $cl(S \cup S') = cl(S) \cup cl(S')$. Every topological closure operator induces a topology: the closed sets are the fixed points of the closure operator. Conversely, every topology $\mathcal{O}(X)$ on X defines a topological closure operator, which maps a subset S of X to the intersection of all closed sets containing S. This closure operator can also be characterized as follows: Let S be a subset of X. An element x in X is a *cluster point* of S if for every open set $o \in \mathcal{O}(X)$, $x \in o$ implies $o \cap (S \setminus \{x\}) \neq \emptyset$; that is, x cannot be separated from S using open sets. Let S^d be the collection of all cluster points of S (it is called the *derived set*). Then

$$cl(S) = S \cup S^d.$$

Let $\langle X, \mathcal{O}(X)\rangle$ be a topological space. A non-empty subset $\mathcal{F} \subseteq \mathcal{O}(X)$ is a *filter* if it satisfies

1. if $o_1 \in \mathcal{F}$ and $o_1 \subseteq o_2$ then $o_2 \in \mathcal{F}$; and
2. if $o_1 \in \mathcal{F}$ and $o_2 \in \mathcal{F}$ then $o_1 \cap o_2 \in \mathcal{F}$.

For instance, every element x in X induces a filter $\mathcal{N}(x) = \{o \in \mathcal{O}(X) \mid x \in o\}$. More generally, any sequence $(x_n)_n$ in X induces a filter

$$\mathcal{N}((x_n)_n) = \{o \in \mathcal{O}(X) \mid \exists N \geq 0 \, \forall n \geq N, \;\; x_n \in o\}.$$

A filter $\mathcal{F}$ *converges* to an element x, denoted by $\mathcal{F} \rightarrow x$, if $\mathcal{N}(x) \subseteq \mathcal{F}$. A sequence $(x_n)_n$ is said to converge to an element x if $\mathcal{N}((x_n)_n) \rightarrow x$.

A function $f : X \rightarrow Y$ between two topological spaces X and Y is *topologically continuous* if the inverse image $f^{-1}(o) = \{x \in X \mid f(x) \in o\}$ of any o in $\mathcal{O}(Y)$ is in $\mathcal{O}(X)$. If $f : X \rightarrow Y$ is topologically continuous then for every sequence $(x_n)_n$ in X and $x \in X$

$$\mathcal{N}((x_n)_n) \rightarrow x \quad \Rightarrow \quad (\mathcal{N}((f(x_n)_n) \rightarrow f(x).$$

The standard topology associated with an ordinary (ultra)metric space X is the *ϵ-ball* topology: a set $o \subseteq X$ is open if

$$\forall x \in o \, \exists \epsilon > 0, \;\; B_\epsilon(x) \subseteq o,$$

where $B_\epsilon(x) = \{y \in X \mid X(x,y) < \epsilon\}$. The set $\{B_\epsilon(x) \mid x \in X \;\&\; \epsilon > 0\}$ is a basis for *ϵ-ball* topology.

The standard topology associated with a preorder X is the *Alexandroff* topology, for which a set $o \subseteq X$ is open if, for x and y in X,

$$x \in o \text{ and } x \leq y \;\Rightarrow\; y \in o,$$

that is, o is upper-closed. If the preorder has a least upper bound for every ω-chain, then the *Scott* topology is more appropriate. It consists of those upper closed subsets $o \subseteq X$ that moreover satisfy, for any ω-chain $(x_n)_n$ in X,

$$\bigsqcup x_n \in o \;\Rightarrow\; \exists N \; \forall n \geq N, \;\; x_n \in o.$$

Clearly, every Scott open set is also Alexandroff open. The converse is generally not true if the pre-order X is not finite. If X is an ω-algebraic cpo with basis B_X then the set $\{\uparrow b \mid b \in B_X\}$, with $\uparrow b = \{x \in X \mid b \leq x\}$, is a basis for the Scott topology.

B Sequences of sequences

The following two lemmas express that the limit of a Cauchy sequence which consists of the limits of Cauchy sequences of finite elements, can be obtained as the limit of a (kind of) diagonal sequence of finite elements.

Lemma B.1 *Let X be a subspace of a complete quasi ultrametric space Y. Let all elements of X be finite in Y. For every $n \geq 0$ let $(v_n^m)_m$ be a Cauchy sequence in X with limit*

$$\lim\nolimits_m v_n^m = y_n. \tag{16}$$

Assume that $(y_n)_n$ is a Cauchy sequence in Y satisfying

$$\forall n : Y\,(y_n, y_{n+1}) \leq \tfrac{1}{3n}. \tag{17}$$

Then there exist subsequences $(x_n^m)_m$ of $(v_n^m)_m$ in X satisfying

$$\forall m : \forall n : X\,(x_n^m, x_{n+1}^m) \leq \tfrac{1}{n} \tag{18}$$

$$\forall n : \forall m : X\,(x_n^m, x_n^{m+1}) \leq \tfrac{1}{m} \tag{19}$$

$$\forall n : \lim\nolimits_m x_n^m = y_n \tag{20}$$

$$\begin{tikzcd}
y_1 \arrow[r, "\frac{1}{3}"] & y_2 \arrow[r, "\frac{1}{6}"] & y_3 \arrow[r, "\frac{1}{9}"] & \cdots \arrow[r] & \lim_n y_n \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \\
x_1^3 \arrow[u, "\frac{1}{3}"] \arrow[r, "1"] & x_2^3 \arrow[u, "\frac{1}{3}"] \arrow[r, "\frac{1}{2}"] & x_3^3 \arrow[u, "\frac{1}{3}"] \arrow[r, "\frac{1}{3}"] & \cdots & \\
x_1^2 \arrow[u, "\frac{1}{2}"] \arrow[r, "1"] & x_2^2 \arrow[u, "\frac{1}{2}"] \arrow[r, "\frac{1}{2}"] & x_3^2 \arrow[u, "\frac{1}{2}"] \arrow[r, "\frac{1}{3}"] & \cdots & \\
x_1^1 \arrow[u, "1"] \arrow[r, "1"] & x_2^1 \arrow[u, "1"] \arrow[r, "\frac{1}{2}"] & x_3^1 \arrow[u, "1"] \arrow[r, "\frac{1}{3}"] & \cdots &
\end{tikzcd}$$

Proof Without loss of generality we can assume that

$$\forall n : \forall m : X\,(v_n^m, v_n^{m+1}) \leq \tfrac{1}{m}.$$

We will construct subsequences $(x_n^m)_m$ of $(v_n^m)_m$ satisfying (18). Because a subsequence of a Cauchy sequence is again Cauchy and has the same limit, these subsequences also satisfy (19) and (20).

Since, for all n,

$$\begin{array}{l}\underleftarrow{\lim}_m Y\left(v_n^m, y_{n+1}\right)\\ \quad = \quad Y\left(\lim_m v_n^m, y_{n+1}\right)\\ \quad = \quad Y\left(y_n, y_{n+1}\right) \quad [(16)]\\ \quad \leq \quad \frac{1}{3n} \quad [(17)]\end{array}$$

we can conclude, according to Proposition 3.1 that

$$\forall n : \exists M_n : \forall m \geq M_n : Y\left(v_n^m, y_{n+1}\right) \leq \tfrac{2}{3n}.$$

By removing from each sequence $(v_n^m)_m$ the first M_n elements we obtain the subsequences $(w_n^m)_m = (v_n^{M_n+m})_m$ satisfying

$$\forall n : \forall m : Y\left(w_n^m, y_{n+1}\right) \leq \tfrac{2}{3n}. \tag{21}$$

Since, for all n and m,

$$\begin{array}{l}\lim_k Y\left(w_n^m, w_{n+1}^k\right)\\ \quad = \quad Y\left(w_n^m, \lim_k w_{n+1}^k\right) \quad [w_n^m \text{ is finite in } Y]\\ \quad = \quad Y\left(w_n^m, y_{n+1}\right) \quad [(16)]\\ \quad \leq \quad \frac{2}{3n} \quad [(21)]\end{array}$$

we have, according to Proposition 3.1 that

$$\forall n : \forall m : \exists K_n^m : \forall k \geq K_n^m : Y\left(w_n^m, w_{n+1}^k\right) \leq \tfrac{1}{n}.$$

Without loss of generality we can assume that the sequences $(K_n^m)_m$ are strictly increasing. The subsequences $(x_n^m)_m = (w_n^{L_n^m})_m$ where

$$L_n^m = \begin{cases} m & \text{if } n = 1\\ K_{n-1}^{L_{n-1}^m} & \text{if } n > 1\end{cases}$$

satisfy (18). □

The above proof shows some resemblance with the proof of Theorem 2 of [Smy87]. The completeness of Y ensures the existence of the limits of the Cauchy sequences $(v_n^m)_m$. If we drop the condition that all elements of X are finite in Y, then the above lemma does not hold any more.

Lemma B.2 *Let X be a subspace of a complete quasi ultrametric space Y. Let $(y_n)_n$ be a Cauchy sequence in Y satisfying*

$$\forall n : Y\left(y_n, y_{n+1}\right) \leq \tfrac{1}{3n}. \tag{22}$$

Let $(x_n^m)_m$ be Cauchy sequences in X satisfying

$$\forall m : \forall n : X\left(x_n^m, x_{n+1}^m\right) \leq \tfrac{1}{n} \tag{23}$$
$$\forall n : \forall m : X\left(x_n^m, x_n^{m+1}\right) \leq \tfrac{1}{m} \tag{24}$$
$$\forall n : \lim_m x_n^m = y_n \tag{25}$$

Then $(x_k^k)_k$ is a Cauchy sequence in X and $\lim_k x_k^k = \lim_n y_n$.

$$
\begin{tikzcd}
y_1 \arrow[r, "\frac{1}{3}"] & y_2 \arrow[r, "\frac{1}{6}"] & y_3 \arrow[r, "\frac{1}{9}"] & \cdots \arrow[r] & \lim_k x_k^k = \lim_n y_n \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \\
x_1^3 \arrow[r, "1"] \arrow[u, "\frac{1}{3}"] & x_2^3 \arrow[r, "\frac{1}{2}"] \arrow[u, "\frac{1}{3}"] & x_3^3 \arrow[r, "\frac{1}{3}"] \arrow[u, "\frac{1}{3}"] \arrow[uurr, dashed] & \cdots & \\
x_1^2 \arrow[r, "1"] \arrow[u, "\frac{1}{2}"] & x_2^2 \arrow[r, "\frac{1}{2}"] \arrow[u, "\frac{1}{2}"] \arrow[ur, dashed] & x_3^2 \arrow[r, "\frac{1}{3}"] \arrow[u, "\frac{1}{2}"] & \cdots & \\
x_1^1 \arrow[r, "1"] \arrow[u, "1"] \arrow[ur, dashed] & x_2^1 \arrow[r, "\frac{1}{2}"] \arrow[u, "1"] & x_3^1 \arrow[r, "\frac{1}{3}"] \arrow[u, "1"] & \cdots &
\end{tikzcd}
$$

Proof Because, for all k,

$$
\begin{aligned}
& X\,(x_k^k, x_{k+1}^{k+1}) \\
\leq \quad & \max\{X\,(x_k^k, x_{k+1}^k), X\,(x_{k+1}^k, x_{k+1}^{k+1})\} \\
\leq \quad & \tfrac{1}{k} \quad [(23) \text{ and } (24)]
\end{aligned}
$$

the sequence $(x_k^k)_k$ is Cauchy.

For all n, m, and k, with $k \geq \max\{n, m\}$,

$$
\begin{aligned}
& Y\,(x_n^m, x_k^k) \\
\leq \quad & \max\{Y\,(x_n^m, x_n^k), Y\,(x_n^k, x_k^k)\} \\
\leq \quad & \max\{\tfrac{1}{m}, \tfrac{1}{n}\} \quad [(23) \text{ and } (24)]
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& Y\,(\lim_n y_n, \lim_k x_k^k) \\
= \quad & \underleftarrow{\lim}_n\, Y\,(y_n, \lim_k x_k^k) \\
= \quad & \underleftarrow{\lim}_n\, Y\,(\lim_m x_n^m, \lim_k x_k^k) \quad [(25)] \\
= \quad & \underleftarrow{\lim}_n\, \underleftarrow{\lim}_m\, Y\,(x_n^m, \lim_k x_k^k) \\
\leq \quad & \underleftarrow{\lim}_n\, \underleftarrow{\lim}_m\, \lim_k Y\,(x_n^m, x_k^k) \quad [\text{Proposition 3.5}] \\
\leq \quad & \underleftarrow{\lim}_n\, \underleftarrow{\lim}_m\, \lim_k \max\{\tfrac{1}{m}, \tfrac{1}{n}\} \quad [\text{see above}] \\
= \quad & 0 \quad [\text{Proposition 3.1}].
\end{aligned}
$$

For all n, m, and k, with $n \geq k$ and $m \geq k$,

$$
\begin{aligned}
& Y\,(x_k^k, x_n^m) \\
\leq \quad & \max\{Y\,(x_k^k, x_k^m), Y\,(x_k^m, x_n^m)\} \\
\leq \quad & \tfrac{1}{k} \quad [(23) \text{ and } (24)]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& Y\,(\lim_k x_k^k, \lim_n y_n) \\
= \quad & \underleftarrow{\lim}_k\, Y\,(x_k^k, \lim_n y_n) \\
\leq \quad & \underleftarrow{\lim}_k\, \lim_n Y\,(x_k^k, y_n) \quad [\text{Proposition 3.5}]
\end{aligned}
$$

$$
\begin{aligned}
&= \varprojlim_k \lim_n Y\,(x_k^k, \lim_m x_n^m) \quad [(25)] \\
&\leq \varprojlim_k \lim_n \lim_m Y\,(x_k^k, x_n^m) \quad [\text{Proposition 3.5}] \\
&\leq \varprojlim_k \lim_n \lim_m \tfrac{1}{k} \quad [\text{see above}] \\
&= 0 \quad [\text{Proposition 3.1}].
\end{aligned}
$$

From the above we can conclude that $\lim_k x_k^k = \lim_n y_n$. $\square$

From the above two lemmas we can conclude the following.

Proposition B.3 *Let X be a subspace of a complete quasi ultrametric space Y. Let all elements of X be finite in Y. Then*

$$\{\ \lim x_n \mid (x_n)_n \textit{ is a Cauchy sequence in } X\ \}$$

is a complete subspace of Y.

Proof Let $R = \{\ \lim x_n \mid (x_n)_n$ is a Cauchy sequence in $X\ \}$. Clearly R is a subspace of Y. Let $(y_n)_n$ be a Cauchy sequence in R. We have to show that its limit $\lim_n y_n$ is an element of R. Without loss of generality we can assume that $\forall n : Y\,(y_n, y_{n+1}) \leq \frac{1}{3n}$. From Lemma B.1 and B.2 we can conclude that there exists a Cauchy sequence $(x_k^k)_k$ in X satisfying $\lim_k x_k^k = \lim_n y_n$. Consequently, $\lim_n y_n \in R$. $\square$