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Solving Domain Equations in a Category of Compact Metric Spaces

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Solving Domain Equations in a Category of Compact Metric Spaces

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Abstract

In order to solve domain equations over 1-bounded compact metric spaces, fixed points of functors on categories of 1-bounded compact metric spaces are studied. Two categories of 1-bounded compact metric spaces are considered: KMS and KMS^E. In both categories, objects are isomorphic if and only if they are isometric. As a consequence, provided that the operation of a domain equation can be extended to a functor, if the functor has a fixed point then this fixed point is a solution of the domain equation and vice versa.

It is shown that so-called locally contractive functors on KMS and contractive functors on KMS^E have fixed points. Furthermore, it is shown that locally contractive functors on KMS and KMS^E have at most one fixed point (up to isomorphism). Hence, locally contractive functors on KMS and contractive and locally contractive functors on KMS^E have unique fixed points.

Examples are presented of extensions of various operations to functors, a simple operation which cannot be extended to a functor, and a functor not having a fixed point.

Most of the results in this report are based on similar - already known - results for 1-bounded complete metric spaces.

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Introduction

In the denotational approach to programming language semantics, *domains* specified by (recursive) *equations* play an important role. As domains, structures like complete partial orders, complete metric spaces, nonwellfounded sets, and wellfounded sets are used. In this report, we will focus on *complete metric spaces*. (For a detailed overview of the use of complete metric spaces in denotational semantics we refer the reader to [BR92].)

Solving domain equations over complete metric spaces has been studied by De Bakker and Zucker ([BZ82]), America and Rutten ([AR89]), Majster-Cederbaum and Zetzsche ([MC88, MC89, MCZ91]), and Rutten and Turi ([RT92]).

In this report, we restrict our attention to *compact metric spaces*. The compactness of a complete metric space is frequently exploited in semantics.

* In [GR83], Golson and Rounds introduce a complete pseudometric space of Milner's rigid synchronization trees ([Mil80]). From the *compactness* of the space, the *compactness theorem* for Hennessy Milner logic ([HM80]) can be derived.

* In [Llo87], Lloyd studies the semantics of nonterminating logic programs. The complete Herbrand universe and base are both turned into a complete metric space. Because of the *compactness* of these spaces, a richer set of properties of the immediate consequence operator can be proved.

* Two denotational semantics for a simple language are proved to be equivalent by De Bakker and Meyer in [BM87]. The first model is order-theoretic and the second model uses a complete metric space. The equivalence proof relies on the *compactness* of the complete metric space used in the second model.

* In [Bre93], Van Breugel proves three complete metric spaces of so-called processes to be isometric. These processes are frequently used in semantics (see, e.g., [BBKM84]) and have the nice property of representing bisimilarity equivalence classes (see [GR89]). In the proof that the three spaces are isometric, the *compactness* of the spaces is exploited.

In order to conclude that a domain defined by a domain equation over complete metric spaces is compact, we will solve the domain equation restricted to compact metric spaces (instead of first solving the domain equation and second proving that the solution is compact). The way we will solve domain equations over compact metric spaces is based on ART: [AR89] and [RT92] (the part on solving domain equations over complete metric spaces of the latter paper being a reconstruction of the former paper along the lines of Smyth and Plotkin's standard work on solving domain equations over complete partial orders [SP82]).

Let F be an operation assigning to each compact metric space another compact metric space. Then $X \cong F(X)$ is a domain equation over compact metric spaces. A solution of the equation is a compact metric space which is isometric - isometry being the natural notion of equivalence of metric spaces - to its F-image. The solution of a domain equation can be viewed as a fixed point of F (up to isometry).

In order to find such a solution we provide the collection of compact metric spaces with some additional structure. We turn this collection of objects into the *category KMS* by defining a collection of arrows between them: *nonexpansive functions*. Also the operation of the domain equation is provided with some additional structure by extending it to a *functor*.

Not every operation on compact metric spaces can be extended to a functor (see Example 3.6). However, various operations including the Cartesian product, the disjoint union, the nonempty and compact power set, and the nonexpansive function space will be shown to be extensible to a functor. As we will see, in order to deal with the nonexpansive function space some additional machinery - including the more involved category KMS^E - has to be introduced.

Since a *fixed point* of a functor F is defined to be an object of the category being isomorphic to its F-image, and objects in the above mentioned categories are isomorphic if and only if they are isometric (i.e. we have chosen the right type of arrows), we can derive the following important equivalence. A domain equation has a solution if and only if the corresponding functor (if it exists) has a fixed point. Furthermore, a fixed point of the functor is a solution of the domain equation, and a solution of the domain equation is a fixed point of the functor (provided that the operation of the domain equation can be extended to a functor).

Not every functor has a fixed point (see Example 4.1). For so-called *locally contractive* functors on KMS and so-called *contractive* functors on KMS^E we will show that they have fixed points (in the Theorems 4.6 and 4.13). Furthermore, we will prove that locally contractive functors on KMS and KMS^E have at most one fixed point up to isomorphism (in Theorem 4.14). Consequently, locally contractive functors on KMS and contractive and locally contractive functors on KMS^E have unique fixed points.

We will introduce a so-called *(locally) Lipschitz coefficient* for a functor and provide simple rules to approximate a functor's (locally) Lipschitz coefficient. This (locally) Lipschitz coefficient is defined in such a way that a functor is (locally) contractive if and only if its (locally) Lipschitz coefficient is smaller than 1.

The main part of this report concerns fixed points of functors on KMS and KMS^E. Besides the work on fixed points of functors on categories of complete metric spaces mentioned above, fixed points of functors in various other related categories have been studied. In his PhD thesis [Mat86], Matthews considers a category of compact agreement spaces - being topologically equivalent to 1-bounded compact ultrametric spaces - and continuous functions. In [Ken87], Kent focuses on a category of pseudo-quasimetric spaces and nonexpansive functions. Edalat and Smyth ([ES92]) use a category of compact metric information systems, which is dual to the category KMS we use. In [Wag94], Wagner studies a category of complete abstract preorders over Ω, with Ω a quantale. For a particular choice for Ω one obtains the category studied by Kent. Wagner also studies compactness in his setting.

Most of the results of this report are obtained from results of [AR89] and [RT92] using some theorems from (metric) topology. To our knowledge, the Examples 3.6 and 4.1, and the Theorems 4.6, 4.13 - the main result of the report - and 4.14 - a generalization of Theorem 5.5 of [RT92] - are new.

In the first section, we present some (nonstandard) notions on metric spaces. In the second section, completeness and compactness of metric spaces are considered. In the third section, we show how various operations on compact metric spaces can be extended to functors. Furthermore, we give an example of an operation which cannot be extended to a functor. In the fourth section, we discuss fixed points of functors. We also develop the additional machinery to deal with the nonexpansive function space. We conclude this section and this report with discussing the (locally) Lipschitz coefficient of a functor.

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1. Mathematical preliminaries

In the rest of this report, the reader is assumed to be familiar with some standard notions of metric spaces. In this first section, only some nonstandard notions of metric spaces, i.e. notions which are not found in the main text of [Eng89], are introduced. The notions of category theory we use in this report are introduced in footnotes. In order not to burden the presentation, we will not present the notions in their most general form, but in a way which suits our purposes best. For more details on categories the reader is referred to [ML71].

In this report, *ϵ-Lipschitz functions* will play an important role.

Definition 1.1 Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called *ϵ-Lipschitz* if, for all x, $x' \in X$,

$$d_Y(f(x), f(x')) \leq \epsilon \cdot d_X(x, x').$$

A function is called *nonexpansive* if it is 1-Lipschitz. A function is called *contractive* if it is ϵ-Lipschitz for some $0 \leq \epsilon < 1$.

In the next definition, several (metrics inducing) operations on 1-bounded metric spaces are presented. These operations are used in domain equations.

Definition 1.2 Let (X, d_X) and (Y, d_Y) be 1-bounded metric spaces. Let Z be a set.

1. For all ϵ, with $0 < \epsilon \leq 1$, a new metric on X is defined by

$$(\epsilon \cdot d)_X(x, x') = \epsilon \cdot d_X(x, x').$$

2. The *Hausdorff metric* on the set of nonempty and compact (with respect to the metric d_X) subsets of X, $\mathcal{P}_{nk}(X)$, is defined by

$$\begin{aligned} d_{\mathcal{P}_{nk}(X)}(A,B) = \max\{ & \sup\{\inf\{d_X(a,b) \mid b \in B\} \mid a \in A\}, \\ & \sup\{\inf\{d_X(b,a) \mid a \in A\} \mid b \in B\}\}. \end{aligned}$$

3. A metric on the Cartesian product of X and Y, $X \times Y$, is defined by

$$d_{X \times Y}((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}.$$

4. A metric on the disjoint union of X and Y, $X + Y$, is defined by

$$d_{X+Y}(z,z') = \begin{cases} d_X(z,z') & \text{if } z \in X \text{ and } z' \in X \\ d_Y(z,z') & \text{if } z \in Y \text{ and } z' \in Y \\ 1 & \text{otherwise} \end{cases}$$

5. A metric on the functions from Z to X, $Z \to X$, is defined by

$$d_{Z \to X}(f,f') = \sup\{d_X(f(z), f'(z)) \mid z \in Z\}.$$

6. A metric on the nonexpansive functions from X to Y, $X \to^1 Y$, is defined by

$$d_{X \to^1 Y}(f,f') = \sup\{d_Y(f(x), f'(x)) \mid x \in X\}.$$

In 2, 5, and 6 of the above definition, the boundedness of the original metrics is needed for the defined metrics to be well-defined. We restrict ourselves to 1-bounded bounded spaces, because considering bounded spaces naturally gives rise to the addition of ∞ to the codomain of a metric (cf. [Law73]) and bring us outside standard (metric) topology.

The 1-bounded metrics introduced above induce operations on 1-bounded metric spaces. For example, the disjoint union of the 1-bounded metric spaces (X, d_X) and (Y, d_Y), $(X, d_X) + (Y, d_Y)$, is the 1-bounded metric space $(X + Y, d_{X+Y})$.

Another operation one encounters frequently in domain equations is taking the set of nonempty and *closed* subsets of a given 1-bounded metric space and endowing it with the induced Hausdorff metric. We do not consider this operation, since on the spaces we are interested in - 1-bounded compact metric spaces - the operation coincides with the second operation of the above definition.

Besides the metric on the Cartesian product introduced in Definition 1.2.3 we will employ another metric on the Cartesian product.

Definition 1.3 Let (X, d_X) and (Y, d_Y) be metric spaces. A metric on the Cartesian product of X and Y, this time denoted by $X \otimes Y$, is defined by

$$d_{X \otimes Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y').$$

We will not use the Cartesian product $\otimes$ in domain equations, since it does not preserve 1-boundedness, but use it to turn function composition into a nonexpansive operation.

We conclude this section with Banach's unique fixed point theorem. The uniqueness of fixed points of certain functors will be proved by means of this theorem.

Theorem 1.4 *Let (X, d_X) be a complete metric space. If $f : X \to X$ is contractive then f has a unique fixed point $\mathit{fix}(f)$.*

Proof See Theorem II.6 of [Ban22]. □

2. From completeness to compactness

Most results of this report on 1-bounded compact metric spaces have already been proved for 1-bounded complete metric spaces. To prove the results for 1-bounded compact metric spaces, the following theorem - due to Fréchet ([Fré10]) - will turn out to be very useful.

Theorem 2.1 *A metric space is compact if and only if it is complete and totally bounded.*

Completeness of a metric space is defined in terms of sequences, viz a metric space (X, d_X) is called *complete* if every Cauchy sequence $(x_n)_n$ in (X, d_X) converges to some x in X. Also compactness and totally boundedness can be expressed using sequences. A subsequence of the sequence $(x_n)_n$ will be denoted by $(x_{s(n)})_n$, where s is a strictly increasing mapping from $\mathbb{N}$ to $\mathbb{N}$.

Theorem 2.2 *A metric space (X, d_X) is compact if and only if for every sequence $(x_n)_n$ in X there exists a converging subsequence $(x_{s(n)})_n$.*

Theorem 2.3 *A metric space (X, d_X) is totally bounded if and only if for every sequence $(x_n)_n$ in X and for every ϵ, with $\epsilon > 0$, there exists a subsequence $(x_{s(n)})_n$ satisfying, for all m and n,*

$$d_X(x_{s(m)}, x_{s(n)}) \leq \epsilon.$$

The operations introduced in Definition 1.2 preserve completeness.

Theorem 2.4 *Let (X, d_X) and (Y, d_Y) be 1-bounded metric spaces. Let Z be a set. If (X, d_X) and (Y, d_Y) are complete, then*

1. *for all ϵ, with $0 < \epsilon \leq 1$, $\epsilon \cdot (X, d_X)$,*
2. *$\mathcal{P}_{nk}(X, d_X)$,*
3. *$(X, d_X) \times (Y, d_Y)$,*
4. *$(X, d_X) + (Y, d_Y)$,*
5. *$Z \to (X, d_X)$, and*
6. *$(X, d_X) \to^1 (Y, d_Y)$*

are complete.

Proof The second case is proved in Lemma 3 of [Kur56]. The other cases are easy to prove. □

Of the operations all but $\to$ preserve totally boundedness.

Theorem 2.5 *Let (X, d_X) and (Y, d_Y) be 1-bounded metric spaces. Let Z be a set. If (X, d_X) and (Y, d_Y) are totally bounded, then*

1. *for all ϵ, with $0 < \epsilon \leq 1$, $\epsilon \cdot (X, d_X)$,*
2. *$\mathcal{P}_{nk}(X, d_X)$,*
3. *$(X, d_X) \times (Y, d_Y)$,*
4. *$(X, d_X) + (Y, d_Y)$, and*
5. *$(X, d_X) \to^1 (Y, d_Y)$*

are totally bounded.

PROOF We only treat the second and the fifth case. The other three cases are easy to prove and are left to the reader.

Let us first deal with the second case. We will show that for all sequences $(A_n)_n$ in $\mathcal{P}_{nk}(X)$ and for all ϵ, with $\epsilon > 0$, there exists a subsequence $(A_{s(n)})_n$ satisfying, for all m and n,

$$d_{\mathcal{P}_{nk}(X)}(A_{s(m)}, A_{s(n)}) \leq \epsilon. \tag{2.1}$$

Having shown this we can conclude that $(\mathcal{P}_{nk}(X), d_{\mathcal{P}_{nk}(X)})$ is totally bounded according to Theorem 2.3.

Let $(A_n)_n$ be a sequence in $\mathcal{P}_{nk}(X)$. Let $\epsilon > 0$. Because (X, d_X) is totally bounded, there exists a finite subset F of X satisfying

$$\forall x \in X : \exists f \in F : d_X(x, f) < \tfrac{\epsilon}{2}. \tag{2.2}$$

Let $(F_n)_n$ be the sequence obtained from $(A_n)_n$ and F by defining

$$F_n = \{\, f \in F \mid \exists a \in A_n : d_X(a, f) < \tfrac{\epsilon}{2} \,\}.$$

Because the set F is finite, there exist only finitely many different F_n's. Consequently, there exists a constant subsequence $(F_{s(n)})_n$ of $(F_n)_n$. In order to prove (2.1), it suffices to prove that, for all n,

$$d_{\mathcal{P}_{nk}(X)}(A_{s(n)}, F_{s(n)}) \leq \tfrac{\epsilon}{2}, \tag{2.3}$$

since then

$$\begin{aligned} & d_{\mathcal{P}_{nk}(X)}(A_{s(m)}, A_{s(n)}) \\ & \quad \leq \quad d_{\mathcal{P}_{nk}(X)}(A_{s(m)}, F_{s(m)}) + d_{\mathcal{P}_{nk}(X)}(F_{s(m)}, F_{s(n)}) + d_{\mathcal{P}_{nk}(X)}(F_{s(n)}, A_{s(n)}) \\ & \quad \leq \quad \tfrac{\epsilon}{2} + 0 + \tfrac{\epsilon}{2}. \end{aligned}$$

We finish the proof of the second case by proving (2.3). From the definition of F_n we can derive that

$$\forall f \in F_{s(n)} : \exists a \in A_{s(n)} : d_X(f, a) < \tfrac{\epsilon}{2}.$$

Consequently,

$$\sup \{\, \inf \{\, d_X(f, a) \mid a \in A_{s(n)} \,\} \mid f \in F_{s(n)} \,\} \leq \tfrac{\epsilon}{2}. \tag{2.4}$$

From (2.2) we can conclude that

$$\forall a \in A_{s(n)} : \exists f \in F : d_X(a, f) < \tfrac{\epsilon}{2}.$$

From the definition of F_n we can derive that $f \in F_n$, and hence

$$\forall a \in A_{s(n)} : \exists f \in F_{s(n)} : d_X(a, f) < \tfrac{\epsilon}{2}.$$

Consequently,

$$\sup \{\, \inf \{\, d_X(a, f) \mid f \in F_{s(n)} \,\} \mid a \in A_{s(n)} \,\} \leq \tfrac{\epsilon}{2}. \tag{2.5}$$

From (2.4) and (2.5) we can conclude (2.3), since

$$\begin{aligned} & d_{\mathcal{P}_{nk}(X)}(A_{s(n)}, F_{s(n)}) \\ & \quad = \quad \max \{ \sup \{\, \inf \{\, d_X(f, a) \mid a \in A_{s(n)} \,\} \mid f \in F_{s(n)} \}, \\ & \qquad\qquad\quad \sup \{\, \inf \{\, d_X(a, f) \mid f \in F_{s(n)} \,\} \mid a \in A_{s(n)} \,\} \} \\ & \quad \leq \quad \max \{ \tfrac{\epsilon}{2}, \tfrac{\epsilon}{2} \}. \end{aligned}$$

Next, we treat the fifth case. As in the previous case, totally boundedness of the space is proved using Theorem 2.3. That is, we will show that for all sequences $(f_n)_n$ in $X \rightarrow^1 Y$ and for all ϵ, with $\epsilon > 0$, there exists a subsequence $(f_{s(n)})_n$ satisfying, for all m and n,

$$d_{X \rightarrow^1 Y}(f_{s(m)}, f_{s(n)}) \leq \epsilon. \tag{2.6}$$

Let $(f_n)_n$ be a sequence in $X \rightarrow^1 Y$. Let $\epsilon > 0$. Because (X, d_X) is totally bounded, there exists a finite subset A of X satisfying

$$\forall x \in X : \exists a \in A : d_X(x, a) < \tfrac{\epsilon}{3}. \tag{2.7}$$

Since $(f_n \restriction A)_n$, where $f_n \restriction A$ denotes the restriction of f_n to A, is a sequence in $A \to Y$ and $A \to Y$ is totally bounded (immediate consequence of the facts that the set A is finite and the metric space (Y, d_Y) is totally bounded), there exists a subsequence $(f_{s(n)} \restriction A)_n$ satisfying, for all m and n,

$$d_{A \to Y}\left(f_{s(m)} \restriction A, f_{s(n)} \restriction A\right) \leq \tfrac{\epsilon}{3} \tag{2.8}$$

according to Theorem 2.3. Consequently, for all m, n, and x,

$$\begin{array}{ll} & d_Y\left(f_{s(m)}(x), f_{s(n)}(x)\right) \\ \leq & d_Y\left(f_{s(m)}(x), f_{s(m)}(a)\right) + d_Y\left(f_{s(m)}(a), f_{s(n)}(a)\right) + d_Y\left(f_{s(n)}(a), f_{s(n)}(x)\right) \\ \leq & d_X(x, a) + d_{A \to Y}\left(f_{s(m)} \restriction A, f_{s(n)} \restriction A\right) + d_X(a, x) \quad [f_{s(m)} \text{ and } f_{s(n)} \text{ are nonexpansive}] \\ \leq & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad [(2.7) \text{ and } (2.8)] \end{array}$$

From the above we can conclude (2.6), since, for all m and n,

$$\begin{array}{ll} & d_{X \to^1 Y}\left(f_{s(m)}, f_{s(n)}\right) \\ = & \sup\left\{ d_Y\left(f_{s(m)}(x), f_{s(n)}(x)\right) \mid x \in X \right\} \\ \leq & \epsilon. \end{array}$$

$\square$

The operation $\to$ does not preserve totally boundedness as is shown in

Example 2.6 Consider the set

$$X = \{ \tfrac{1}{n} \mid n \in \mathbb{N} \} \cup \{0\}$$

endowed with the Euclidean metric. The obtained metric space is totally bounded. Let $f_n : X \to X$ be defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ x & \text{otherwise} \end{cases}$$

Because, for all m and n, with $m \neq n$, we have that

$$d_{X \to X}(f_m, f_n) \geq \tfrac{1}{2},$$

the space $X \to X$ is not totally bounded. Because f_n is continuous, also if we restrict ourselves to continuous functions we do not obtain a totally bounded metric space.

Combining the Theorems 2.1, 2.4, and 2.5 we obtain

Theorem 2.7 *Let (X, d_X) and (Y, d_Y) be 1-bounded metric spaces. Let Z be a set. If (X, d_X) and (Y, d_Y) are compact, then*

1. *for all ϵ, with $0 < \epsilon \leq 1$, $\epsilon \cdot (X, d_X)$,*
2. *$\mathcal{P}_{nk}(X, d_X)$,*
3. *$(X, d_X) \times (Y, d_Y)$,*
4. *$(X, d_X) + (Y, d_Y)$, and*
5. *$(X, d_X) \to^1 (Y, d_Y)$*

are compact.

We conclude this section with some remarks on the above theorem. None of the results listed in the theorem are new. The results can also be proved directly, i.e. without splitting compactness into completeness and totally boundedness. Also in that case, all but the second and the fifth case are easy to prove. A direct proof of the second case is presented in, e.g., Theorem 4.2 of [Mic51]. The fifth case is a consequence of the theorem of Arzelà-Ascoli (see, e.g., page 267 of [Dug66]). We have chosen for splitting compactness into completeness and totally boundedness, since this will turn out to be fruitful in the sequel.

3. From operations to functors

As already mentioned in the introduction, in this report we will focus on fixed points of functors in order to find solutions of domain equations. In this section, we will point out our limitations by showing that, although the operations of Definition 1.2 which preserve compactness can all be extended to a functor, there exists a simple example of an operation which cannot be extended to a functor.

We introduce the *category*[1] *KMS* in

Definition 3.1 The category *KMS* has 1-bounded compact metric spaces as objects and nonexpansive functions as arrows. The domain and codomain of arrows are the domain and codomain of the functions. The composition of arrows is the function composition of the functions. The identity arrows are the identity functions.

Note that in this category objects are *isomorphic*[2] if and only if they are isometric. This is also the case for the closely related category *CMS* used in [AR89] and [RT92].

Definition 3.2 The category *CMS* has 1-bounded complete metric spaces as objects and nonexpansive functions as arrows. The domain and codomain of arrows are the domain and codomain of the functions. The composition of arrows is the function composition of the functions. The identity arrows are the identity functions.

The operations $\epsilon\cdot$ and $\mathcal{P}_{nk}$ introduced in Definition 1.2 are extended to a *functor*[3] by defining how these operations act on arrows.

Definition 3.3 Let (X, d_X) and (Y, d_Y) be 1-bounded compact metric spaces. Let $f \in (X, d_X) \to^1 (Y, d_Y)$.

* For all ϵ, with $0 < \epsilon \leq 1$, the function $\epsilon \cdot f : \epsilon \cdot (X, d_X) \to \epsilon \cdot (Y, d_Y)$ is defined by

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[1] A *category* consists of
- a class of *objects*,
- a class of *arrows*,
- a function *dom* assigning to each arrow an object, called *domain*,
- a function *cod* assigning to each arrow an object, called *codomain*,
- $*$ We will write $f : X \to Y$ to denote that $dom(f) = X$ and $cod(f) = Y$. The arrow f is a so-called *arrow from X to Y*. The set of all arrows from X to Y, called *homset*, is denoted by $hom(X, Y)$. We will call a pair of arrows (f, g) *composable* if $cod(f) = dom(g)$.
- a function $\circ$ assigning to each pair of composable arrows (f, g) an arrow $g \circ f : dom(f) \to cod(g)$, called the *composition* of f and g, which is *associative*, i.e. for all pairs of composable arrows (f, g) and (g, h),
  $h \circ (g \circ f) = (h \circ g) \circ f$,
- $*$ We can also introduce the function $\circ$ in terms of homsets: for all objects X, Y, and Z, a function
  $\circ : hom(Y, Z) \times hom(X, Y) \to hom(X, Z)$.
- for each object X, an arrow $id_X : X \to X$, called the *identity* of X, which satisfies the *unit law*, i.e. for all arrows $f : X \to Y$, we have that
  $f \circ id_X = f$ and $id_Y \circ f = f$.

[2] Let $\mathcal{C}$ be a category. Two objects X and Y are called *isomorphic* if there exist arrows $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$ forming an *isomorphism*, i.e.
$g \circ f = id_X$ and $f \circ g = id_Y$.

[3] Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a function assigning to each object in $\mathcal{C}$ an object in $\mathcal{D}$ and assigning to each arrow $f : X \to Y$ in $\mathcal{C}$ an arrow $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ such that
- it *preserves identities*, i.e. for each object X in $\mathcal{C}$,
  $F(id_X) = id_{F(X)}$,
- it *preserves compositions*, i.e. for all pairs of composable arrows (f, g) in $\mathcal{C}$,
  $F(g \circ f) = F(g) \circ F(f)$.

$\epsilon \cdot f = f.$

* The function $\mathcal{P}_{nk}(f) : \mathcal{P}_{nk}(X, d_X) \to \mathcal{P}(Y, d_Y)$ is defined by

$$\mathcal{P}_{nk}(f)(A) = \{ f(a) \mid a \in A \}.$$

In order to conclude that we have extended the operations to functors, we have to check that if f is an arrow in KMS then $\epsilon \cdot f$ and $\mathcal{P}_{nk}(f)$ are arrows in KMS, and that $\epsilon \cdot$ and $\mathcal{P}_{nk}$ preserve identities and compositions. This is easy and left to the reader (cf. Lemma 5.2 of [AR89]).

Furthermore, for each 1-bounded compact metric space (X, d_X), the operation assigning to each 1-bounded compact metric space the space (X, d_X) can be extended to a functor on KMS: every arrow is assigned to the identity arrow of (X, d_X).

The operations $\times$ and $+$ are extended to functors from the *product category*[4] $KMS \times KMS$ - in the sequel abbreviated to KMS^2 - to the category KMS as follows.

Definition 3.4 Let (U, d_U), (V, d_V), (X, d_X), and (Y, d_Y) be 1-bounded compact metric spaces. Let $f \in (U, d_U) \to^1 (X, d_X)$ and $g : (V, d_V) \to^1 (Y, d_Y)$.

* The function $f \times g : ((U, d_U) \times (V, d_V)) \to ((X, d_X) \times (Y, d_Y))$ is defined by

$$(f \times g)(u, v) = (f(u), g(v)).$$

* The function $f + g : ((U, d_U) + (V, d_V)) \to ((X, d_X) + (Y, d_Y))$ is defined by

$$(f + g)(w) = \begin{cases} f(w) & \text{if } w \in U \\ g(w) & \text{if } w \in V \end{cases}$$

One can easily verify that the operations $\times$ and $+$ map arrows in KMS^2 to arrows in KMS, preserve identities, and preserve compositions.

The operation $\to^1$ can also be extended to a functor. To deal with $\to^1$ we need some additional machinery. We will develop this machinery in the Subsection 4.1.

By means of *composition* and *tupling*[5] we can form new functors from the above defined ones.

Example 3.5 The operation of the domain equation

$$(X, d_X) \cong (Y, d_Y) \times \tfrac{1}{2} \cdot (X, d_X),$$

where (Y, d_Y) is an (arbitrary) 1-bounded compact metric space, can be extended to the functor

$$\times \circ ((Y, d_Y), \tfrac{1}{2} \cdot).$$

The operation of the domain equation

$$(X, d_X) \cong (Y, d_Y) + \mathcal{P}_{nk}((Z, d_Z) \times \tfrac{1}{2} \cdot (X, d_X)),$$

---

[4] Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ has as objects pairs (X, Y), with X an object in $\mathcal{C}$ and Y an object in $\mathcal{D}$, and as arrows pairs $(f, g) : (U, V) \to (X, Y)$, with $f : U \to X$ an arrow in $\mathcal{C}$ and $g : V \to Y$ an arrow in $\mathcal{D}$. The composition of the arrows (f, g) and (h, i) is $(h \circ f, i \circ g)$. The identity arrows are pairs of identity arrows.

[5] Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories. Let $F : \mathcal{C} \to D$, $G : \mathcal{D} \to \mathcal{E}$, and $H : \mathcal{C} \to \mathcal{E}$ be functors.

- The *composition* of G with F is the functor $G \circ F : \mathcal{C} \to \mathcal{E}$ defined by
$$\begin{aligned} (G \circ F)(X) &= G(F(X)) \\ (G \circ F)(f) &= G(F(f)) \end{aligned}$$
- The *tupling* of F and H is the functor $(F, H) : \mathcal{C} \to (\mathcal{D} \times \mathcal{E})$ defined by
$$\begin{aligned} (F, H)(X) &= (F(X), H(X)) \\ (F, H)(f) &= (F(X), H(f)) \end{aligned}$$

where (Y, d_Y) and (Z, d_Z) are (arbitrary) 1-bounded compact metric spaces, can be extended to the functor

$$+ \circ ((Y, d_Y), \mathcal{P}_{nk} \circ \times \circ ((Z, d_Z), \tfrac{1}{2}\cdot)).$$

However, not every operation can be extended to a functor as the following example shows us.

Example 3.6 The operation F assigning to each 1-bounded compact metric space (X, d_X) the 1-bounded compact metric space

$$F(X, d_X) = \begin{cases} diam\,(X, d_X) \cdot (X, d_X) & \text{if } diam\,(X, d_X) > 0 \\ (X, d_X) & \text{otherwise} \end{cases}$$

where

$$diam\,(X, d_X) = \sup_{x,y \in X} d_X(x, y),$$

cannot be extended to a functor as we will see.

Consider the sets $X = \{0, \frac{1}{2}\}$ and $Y = \{0, \frac{1}{2}, 1\}$ endowed with the Euclidean metric. Clearly, these metric spaces are objects in KMS. Consider the functions $f : \{0, \frac{1}{2}\} \to \{0, \frac{1}{2}, 1\}$ defined by

$$\begin{aligned} f(0) &= 0 \\ f(\tfrac{1}{2}) &= \tfrac{1}{2} \end{aligned}$$

and $g : \{0, \frac{1}{2}, 1\} \to \{0, \frac{1}{2}\}$ defined by

$$\begin{aligned} g(0) &= 0 \\ g(\tfrac{1}{2}) &= \tfrac{1}{2} \\ g(1) &= \tfrac{1}{2} \end{aligned}$$

The functions can be depicted by

$$\begin{tikzcd}
 & 1 \arrow[dr] & \\
\frac{1}{2} \arrow[r] & \frac{1}{2} \arrow[r] & \frac{1}{2} \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Obviously, the functions f and g are arrows in KMS. Now assume that the operation F can be extended to a functor. Then $F(f)$ and $F(g)$ are arrows of KMS, i.e. the functions $F(f)$ and $F(g)$ are nonexpansive. Consequently,

$$\begin{aligned} & d_{F(Y)}(F(f)(0), F(f)(\tfrac{1}{2})) \\ \leq\ & d_{F(X)}(0, \tfrac{1}{2}) \quad [F(f) \text{ is nonexpansive}] \\ =\ & \tfrac{1}{2} \cdot d_X(0, \tfrac{1}{2}) \\ =\ & \tfrac{1}{4}. \end{aligned}$$

Since all elements of $F(Y)$ have a distance larger than $\frac{1}{4}$ to each other, we have that

$$F(f)(0) = F(f)(\tfrac{1}{2}) \tag{3.9}$$

A possible choice for $F(f)$ and $F(g)$ is depicted by

$$\begin{tikzcd}
 & 1 \arrow[dddr, dotted] & \\
 & \frac{1}{2} \arrow[ddr, dotted] & \\
\frac{1}{2} \arrow[dr, dotted] & & \frac{1}{2} \\
0 \arrow[r, dotted] & 0 \arrow[r, dotted] & 0
\end{tikzcd}$$

Since F is assumed to be a functor, we have that

$$\begin{array}{ll} F(g) \circ F(f) & \\ = & F(g \circ f) \quad [F \text{ preserves compositions}] \\ = & F(id_X) \\ = & id_{F(X)} \quad [F \text{ preserves identities}] \end{array}$$

However, this is in contradiction with (3.9). Consequently, F cannot be extended to a functor.

Although the operation F of the above example cannot be extended to a functor, the domain equation

$$(X, d_X) \cong F(X, d_X)$$

has several solutions. For example, every nonempty and finite set endowed with the discrete metric is a solution of the above domain equation. Hence, by focusing on fixed points of functors rather than solutions of domain equations, we loose some generality.

4. Fixed points of functors

In this final section we will focus on *fixed points*[6] of functors. First of all, we show that not every functor on KMS has a fixed point.

Example 4.1 Consider the functor $F : KMS \to KMS$ defined by

$$F = + \circ (\mathbb{1}, 1\cdot),$$

where $\mathbb{1}$ is the singleton metric space. Assume that (X, d_X) is a fixed point of F. Then the space (X, d_X) is compact and we have that

$$\begin{array}{rcl} (X, d_X) & \cong & \mathbb{1} + (X, d_X) \\ & \cong & \mathbb{1} + (\mathbb{1} + (X, d_X)) \\ & \cong & \mathbb{1} + (\mathbb{1} + (\mathbb{1} + (X, d_X))) \\ & \vdots & \end{array}$$

Consequently, (X, d_X) contains an unbounded number of elements having distance 1 to each other. This contradicts the fact that (X, d_X) is compact. Hence F has no fixed point.

Next, we will introduce a sufficient condition on functors on KMS for having fixed points. In order to formulate this condition we will consider *CMS-categories* - a natural generalization of categories.

Definition 4.2 A *CMS-category* $\mathcal{C}$ is a category with some additional structure: the homsets are objects in CMS and the compositions are arrows in CMS, that is, for all objects X and Y in $\mathcal{C}$, the homset $hom(X, Y)$ is a 1-bounded complete metric space, and, for all objects X, Y, and Z in $\mathcal{C}$, the composition

$$\circ : (hom(Y, Z) \otimes hom(X, Y)) \to hom(X, Z)$$

is a nonexpansive function.

[6] Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to \mathcal{C}$ be a functor. An object X in $\mathcal{C}$ is called a *fixed point* of F if X is isomorphic to $F(X)$.

The above introduced notion *CMS*-category is consistent with the notion $\mathcal{C}$-category, with $\mathcal{C}$ a so-called monoidal category (see, e.g., page 180 of [ML71]).

The category *KMS* can be turned into a *CMS*-category by endowing the homsets with the metric introduced in Definition 1.2.6.

PROPOSITION 4.3 *KMS is a CMS-category.*

PROOF The fact that the homsets of *KMS* are objects in *CMS* is an immediate consequence of Theorem 2.4.6. Let (f, g) and (h, i) be pairs of composable arrows in *KMS* with $f,\, h : (X, d_X) \to (Y, d_Y)$ and $g,\, i : (Y, d_Y) \to (Z, d_Z)$. Then

$$\begin{aligned}
& d(g \circ f, i \circ h) \\
&= \sup_{x \in X} d_Z(g(f(x)), i(h(x))) \\
&\leq \sup_{x \in X} d_Z(g(f(x)), g(h(x))) + d_Z(g(h(x)), i(h(x))) \\
&\leq \sup_{x \in X} d_Z(g(f(x)), g(h(x))) + \sup_{x \in X} d_Z(g(h(x)), i(h(x))) \\
&\leq \sup_{x \in X} d_Y(f(x), h(x)) + \sup_{y \in Y} d_Z(g(y), i(y)) \quad [g \text{ is nonexpansive}] \\
&= d(f, h) + d(g, i) \\
&= d((f, g), (h, i)).
\end{aligned}$$

□

Note that if we replace the $\otimes$ in Definition 4.2 by $\times$, then *KMS* is not a *CMS*-category any more.

If $\mathcal{C}$ and $\mathcal{D}$ are *CMS*-categories, then the product category $\mathcal{C} \times \mathcal{D}$, the homsets of which are endowed with the product (as introduced in Definition 1.2.3) of the metrics of the homsets of $\mathcal{C}$ and $\mathcal{D}$, is a *CMS*-category. Consequently, KMS^2 is a *CMS*-category.

For functors on *CMS*-categories we can introduce the notion of being *locally ϵ-Lipschitz*.

DEFINITION 4.4 Let $\mathcal{C}$ and $\mathcal{D}$ be *CMS*-categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *locally ϵ-Lipschitz* if, for all objects X and Y in $\mathcal{C}$, the function

$$F \restriction hom(X, Y)$$

is ϵ-Lipschitz. A functor is called *locally nonexpansive* if it is locally 1-Lipschitz. A functor is called *locally contractive* if it is locally ϵ-Lipschitz for some $0 \leq \epsilon < 1$. A functor is called *locally Lipschitz* if it is locally ϵ-Lipschitz for some ϵ.

EXAMPLE 4.5 The functor $F : KMS \to KMS$ defined by

$$F = \times \circ ((X, d_X), \tfrac{1}{2} \cdot),$$

where (X, d_X) is an (arbitrary) 1-bounded complete metric spaces, is locally $\frac{1}{2}$-Lipschitz, since for all (Y, d_Y), (Z, d_Z) in *KMS*, and $f,\, g \in hom((Y, d_Y), (Z, d_Z))$, we have that

$$\begin{aligned}
& d(F(f), F(g)) \\
&= \sup\{\, d_{X \times \frac{1}{2} \cdot Z}((x, f(y)), (x, g(y))) \mid (x, y) \in X \times Y \,\} \\
&= \sup\{\, \max\{d_X(x, x), \tfrac{1}{2} \cdot d_Z(f(y), g(y))\} \mid (x, y) \in X \times Y \,\} \\
&= \sup\{\, \tfrac{1}{2} \cdot d_Z(f(y), g(y)) \mid y \in Y \,\} \\
&= \tfrac{1}{2} \cdot d(f, g).
\end{aligned}$$

Every functor on *KMS* which is locally contractive has a fixed point.

THEOREM 4.6 *Let $F : KMS \to KMS$ be a functor. If F is locally contractive then it has a fixed point.*

We will provide the reader with a proof of this theorem at the end of Subsection 4.1. The proof makes use of another - to be presented - fixed point theorem (viz Theorem 4.13). A direct proof of the above theorem would be similar to the proof of this other fixed point theorem.

Note that the condition of being locally contractive is not necessary for having fixed points, since the identity functor $1 \cdot$ is not locally contractive but every object is a fixed point of this functor.

In Subsection 4.2, we will strengthen the above result by showing that the fixed point of a locally contractive functor on KMS is even unique.

4.1 The nonexpansive function space

Next, we will discuss how to extend the nonexpansive function space to a functor. Assume that we try to extend the nonexpansive function space to a functor from KMS^2 to KMS. Let f be an arrow from (U, d_U) to (X, d_X) in KMS and let g be an arrow from (V, d_V) to (Y, d_Y) in KMS. Then $f \rightarrow^1 g$ should be an arrow from $(U, d_U) \rightarrow^1 (V, d_V)$ to $(X, d_X) \rightarrow^1 (Y, d_Y)$ in KMS, i.e. a nonexpansive function assigning to each arrow h from (U, d_U) to (V, d_V) in KMS an arrow $(f \rightarrow^1 g)(h)$ from (X, d_X) to (Y, d_Y) in KMS, as depicted in the following picture.

$$\begin{tikzcd}
(U, d_U) \arrow[r, "h"] \arrow[d, "f"'] & (V, d_V) \arrow[d, "g"] \\
(X, d_X) \arrow[r, dotted, "(f \rightarrow^1 g)(h)"'] & (Y, d_Y)
\end{tikzcd}$$

In this case, it is not clear how to define the arrow $(f \rightarrow^1 g)(h)$. However, if we use the *opposite category*[7] KMS^{op}, then we can extend the nonexpansive function space to a functor from $KMS^{op} \times KMS$ to KMS by defining $(f \rightarrow^1 g)(h)$ as follows.

$$\begin{tikzcd}
(U, d_U) \arrow[r, "h"] & (V, d_V) \arrow[d, "g"] \\
(X, d_X) \arrow[u, "f"] \arrow[r, dotted, "(f \rightarrow^1 g)(h)"'] & (Y, d_Y)
\end{tikzcd}$$

Definition 4.7 Let (U, d_U), (V, d_V), (X, d_X), and (Y, d_Y) be 1-bounded compact metric spaces. Let $f \in (X, d_X) \rightarrow^1 (U, d_U)$ and $g : (V, d_V) \rightarrow^1 (Y, d_Y)$. The function

$$f \rightarrow^1 g : ((U, d_U) \rightarrow^1 (V, d_V)) \rightarrow ((X, d_X) \rightarrow^1 (Y, d_Y))$$

is defined by

$$(f \rightarrow^1 g)(h) = g \circ h \circ f.$$

One can easily verify that the above definition extends the nonexpansive function space to a functor. Because the functor $\rightarrow^1$ uses arrows in "both directions", i.e. arrows in KMS^{op} and KMS, in order to solve domain equations like

$$(X, d_X) \cong (X, d_X) \rightarrow^1 (X, d_X) \tag{4.10}$$

[7] Let $\mathcal{C}$ be a category. The *opposite category* $\mathcal{C}^{op}$ has the objects of $\mathcal{C}$ as objects and the opposites of the arrows of $\mathcal{C}$ as arrows. That is, if $f : X \rightarrow Y$ is an arrow in $\mathcal{C}$ then $f : Y \rightarrow X$ is an arrow in $\mathcal{C}^{op}$. Composition and identities are defined in the obvious way.

we introduce a category of 1-bounded compact metric spaces with arrows in "both directions". (An alternative route for solving domain equations incorporating the function space is taken by Freyd in [Fre90].)

DEFINITION 4.8 Let $\mathcal{C}$ be a category. The category $\mathcal{C}^E$ has as objects the objects of $\mathcal{C}$ and as arrows pairs $(e,p) : X \to Y$ with $e : X \to Y$ and $p : Y \to X$ arrows in $\mathcal{C}$ satisfying $p \circ e = id_X$. The composition of (e,p) and (f,q) is $(f \circ e, q \circ p)$. The identity arrows are pairs of identity arrows.

The category KMS^E has arrows in "both directions". The additional condition that $p \circ e = id_X$ is a technicality we inherit from [AR89].

As in KMS, also in KMS^E objects are isomorphic if and only if they are isometric.

In order to solve equations like (4.10), we will extend the functor $\to^1$ to a functor $\to^{1^E}$ on KMS^E. More generally, every functor from $(\mathcal{C}^{op})^m \times \mathcal{C}^n$ - denoted by $\mathcal{C}^{m,n}$ in the sequel - to $\mathcal{C}$ will be extended to a functor from $(\mathcal{C}^E)^{m+n}$ to $\mathcal{C}^E$.

DEFINITION 4.9 Let $\mathcal{C}$ be a category. Let $F : \mathcal{C}^{m,n} \to \mathcal{C}$ be a functor. The functor

$$F^E : (\mathcal{C}^E)^{m+n} \to \mathcal{C}^E$$

is defined on objects by

$$F^E(X_1, \ldots, X_{m+n}) = F(X_1, \ldots, X_{m+n})$$

and on arrows by

$$\begin{aligned} & F^E((e_1,p_1), \ldots, (e_{m+n}, p_{m+n})) \\ = \quad & (F(p_1, \ldots, p_m, e_{m+1}, \ldots, e_{m+n}), F(e_1, \ldots, e_m, p_{m+1}, \ldots, p_{m+n})). \end{aligned}$$

EXAMPLE 4.10 With the domain equation

$$(X, d_X) \cong (Y, d_Y) + \tfrac{1}{3} \cdot ((X, d_X) \to^1 (X, d_X)),$$

with (Y, d_Y) an (arbitrary) 1-bounded compact metric space, we associate the functor

$$+^E \circ ((Y, d_Y)^E, \tfrac{1}{3} \cdot^E \circ \to^{1^E} \circ (1 \cdot^E, 1 \cdot^E)).$$

Not every functor on KMS^E has a fixed point. As in Example 4.1, one can easily verify that the functor

$$+^E \circ (\mathbb{1}^E, 1 \cdot^E), \tag{4.11}$$

with $\mathbb{1}$ the singleton metric space, has no fixed point.

Next, we will introduce a sufficient condition for functors on KMS^E for having fixed points. Again we will exploit the fact that the category involved - the category KMS^E - can be turned into a CMS-category. (If the category $\mathcal{C}$ is a CMS-category then the category $\mathcal{C}^E$, the homsets of which are endowed with the product (as defined in Definition 1.2.3) of the metric of the homsets of $\mathcal{C}$, is a CMS-category.)

DEFINITION 4.11 Let $\mathcal{C}$ be a CMS-category. A functor $F : (\mathcal{C}^E)^m \to (\mathcal{C}^E)^n$ is called *ϵ-Lipschitz* if, for all arrows $((e_1,p_1), \ldots, (e_m,p_m)) : (X_1, \ldots, X_m) \to (Y_1, \ldots, Y_m)$ in $(\mathcal{C}^E)^m$,

$$\delta(F((e_1,p_1), \ldots, (e_m,p_m))) \leq \epsilon \cdot \delta((e_1,p_1), \ldots, (e_m,p_m)),$$

where

$$\delta((e_1,p_1), \ldots, (e_m,p_m)) = \max\{\, d(e_i \circ p_i, id_{Y_i}) \mid 1 \leq i \leq m \,\}.$$

A functor is called *nonexpansive* if it is 1-Lipschitz. A functor is called *contractive* if it is ϵ-Lipschitz for some $0 \leq \epsilon < 1$. A functor is called *Lipschitz* if it is ϵ-*Lipschitz* for some ϵ.

Every locally ϵ-Lipschitz functor is extended to a ϵ-Lipschitz and locally ϵ-Lipschitz functor by Definition 4.9.

PROPOSITION 4.12 *Let $\mathcal{C}$ be a CMS-category. Let $F : \mathcal{C}^{m,n} \to \mathcal{C}$ be a functor. If F is locally ϵ-Lipschitz then F^E is ϵ-Lipschitz and locally ϵ-Lipschitz.*

PROOF Let $F : \mathcal{C}^{m,n} \to \mathcal{C}$ be a locally ϵ-Lipschitz functor. The fact that the functor F^E is ϵ-Lipschitz is proved in Theorem 5.22 of [RT92]. The functor F^E is also locally ϵ-Lipschitz, because for all $((e_1, p_1), \ldots, (e_{m+n}, p_{m+n}))$, $((f_1, q_1), \ldots, (f_{m+n}, q_{m+n})) : (X_1, \ldots, X_{m+n}) \to (Y_1, \ldots, Y_{m+n})$ we have that

$$\begin{aligned}
&d\,(F^E\,((e_1,p_1),\ldots,(e_{m+n},p_{m+n})), F^E\,((f_1,q_1),\ldots,(f_{m+n},q_{m+n})))\\
&\quad = \quad d\,((F\,(p_1,\ldots,p_m,e_{m+1},\ldots,e_{m+n}), F\,(e_1,\ldots,e_m,p_{m+1},\ldots,p_{m+n})),\\
&\qquad\qquad (F\,(q_1,\ldots,q_m,f_{m+1},\ldots,f_{m+n}), F\,(f_1,\ldots,f_m,q_{m+1},\ldots,q_{m+n})))\\
&\quad = \quad \max\{d\,(F\,(p_1,\ldots,p_m,e_{m+1},\ldots,e_{m+n}), F\,(q_1,\ldots,q_m,f_{m+1},\ldots,f_{m+n})),\\
&\qquad\qquad d\,(F\,(e_1,\ldots,e_m,p_{m+1},\ldots,p_{m+n}), F\,(f_1,\ldots,f_m,q_{m+1},\ldots,q_{m+n}))\}\\
&\quad \leq \quad \max\{\epsilon\cdot d\,((p_1,\ldots,p_m,e_{m+1},\ldots,e_{m+n}),(q_1,\ldots,q_m,f_{m+1},\ldots,f_{m+n})),\\
&\qquad\qquad \epsilon\cdot d\,((e_1,\ldots,e_m,p_{m+1},\ldots,p_{m+n}),(f_1,\ldots,f_m,e_{m+1},\ldots,e_{m+n}))\}\\
&\quad [F \text{ is locally } \epsilon\text{-Lipschitz}]\\
&\quad = \quad \epsilon\cdot d\,(((e_1,p_1),\ldots,(e_{m+n},p_{m+n})),((f_1,q_1),\ldots,(f_{m+n},q_{m+n}))).
\end{aligned}$$

□

Functors on KMS^E which are contractive have a fixed point.

THEOREM 4.13 *Let $F : KMS^E \to KMS^E$ be a functor. If F is contractive then it has a fixed point.*

PROOF Let $F : KMS^E \to KMS^E$ be a ϵ-Lipschitz functor, for some $0 \leq \epsilon < 1$. In order to conclude that F has a fixed point, we will construct an object (X, d_X) in KMS^E which can be shown to be isomorphic to $F\,(X, d_X)$.

First, we will define the above mentioned object (X, d_X). Let (X_1, d_{X_1}) be the singleton metric space and let (e_1, p_1) be an arbitrary arrow from (X_1, d_{X_1}) to $F\,(X_1, d_{X_1})$. A so-called ω-*chain* $\Delta = ((X_n, d_{X_n}), (e_n, p_n))_n$ is constructed by defining, for all n, $(X_{n+1}, d_{X_{n+1}}) = F\,(X_n, d_{X_n})$ and $(e_{n+1}, p_{n+1}) = F\,(e_n, p_n)$. The so-called *direct limit* of Δ is the set X defined by

$$X = \{\,(x_n)_n \mid x_n \in X_n \wedge p_n\,(x_{n+1}) = x_n\,\}$$

endowed with the metric d_X defined by

$$d_X\,((x_n)_n, (y_n)_n) = \sup\{\,d_{X_n}\,(x_n, y_n) \mid n \in \mathbb{N}\,\}.$$

The above supremum exists, since, for all n, the metric d_{X_n} is 1-bounded. Note that

$$\text{the sequence } (d_{X_n}\,(x_n, y_n))_n \text{ is increasing,} \tag{4.12}$$

since

$$\begin{aligned}
&d_{X_n}\,(x_n, y_n)\\
&\quad = \quad d_{X_n}\,(p_n\,(x_{n+1}), p_n\,(y_{n+1}))\\
&\quad \leq \quad d_{X_{n+1}}\,(x_{n+1}, y_{n+1}) \quad [p_n \text{ is nonexpansive}]
\end{aligned}$$

Second, we will show that (X, d_X) is an object in KMS^E. In Lemma 3.10 of [AR89], it is shown that the direct limit of an ω-chain in the category CMS^E is an object in CMS^E, that is, the direct limit is a 1-bounded complete metric space. Clearly, the above defined ω-chain Δ is also an ω-chain in CMS^E. Consequently, (X, d_X) is a 1-bounded complete metric space. In order to conclude that (X, d_X) is an object in KMS^E, that is, (X, d_X) is a 1-bounded compact metric space, we only have to show that (X, d_X) is totally bounded due to Theorem 2.1. According to Theorem 2.3, it suffices to prove that for all sequences $(\bar{x}_n)_n$, with $\bar{x}_n = (x_{n,k})_k$, in X and for all γ, with $\gamma > 0$, there exists a subsequence $(\bar{x}_{s(n)})_n$ satisfying, for all m and n,

$$d_X\left(\bar{x}_{s(m)}, \bar{x}_{s(n)}\right) \leq \gamma.$$

Let $(\bar{x}_n)_n$ be a sequence in X. Let $\gamma > 0$. Because F is ϵ-Lipschitz, the ω-chain Δ is *Cauchy*, i.e.

$$\forall \beta > 0 : \exists N \in \mathbb{N} : \forall m > n \geq N : \delta\left(e_{m,n}, p_{n,m}\right) \leq \beta,$$

where

$$\begin{aligned} e_{m,n} &= e_n \circ \cdots \circ e_{m-1} \\ p_{n,m} &= p_{m-1} \circ \cdots \circ p_n \end{aligned}$$

since

$$\begin{aligned} &\delta\left(e_{m,n}, p_{n,m}\right) \\ &\quad = \quad \delta\left(F^{n-1}\left(e_{m-n+1,1}, p_{1,m-n+1}\right)\right) \quad [F \text{ preserves compositions}] \\ &\quad \leq \quad \epsilon^{n-1} \cdot \delta\left(e_{m-n+1,1}, p_{1,m-n+1}\right) \quad [F \text{ is } \epsilon\text{-Lipschitz}] \\ &\quad \leq \quad \epsilon^{n-1} \quad [d \text{ is 1-bounded}] \end{aligned}$$

Consequently,

$$\exists N \in \mathbb{N} : \forall m > n \geq N : \delta\left(e_{m,n}, p_{n,m}\right) \leq \tfrac{\gamma}{3}. \tag{4.13}$$

Because (X_N, d_{X_N}) is compact, (X_N, d_{X_N}) is totally bounded due to Theorem 2.1. By Theorem 2.3, there exists a subsequence $(x_{s(n),N})_n$ of $(x_{n,N})_n$ satisfying, for all m and n,

$$d_{X_N}\left(x_{s(m),N}, x_{s(n),N}\right) \leq \tfrac{\gamma}{3}. \tag{4.14}$$

For all m, n, and k, with $k > N$, we have that

$$d_{X_k}\left(x_{s(m),k}, x_{s(n),k}\right) \leq \gamma \tag{4.15}$$

since

$$\begin{aligned} &d_{X_k}\left(x_{s(m),k}, x_{s(n),k}\right) \\ &\quad \leq \quad d_{X_k}\left(x_{s(m),k}, \left(e_{N,k} \circ p_{k,N}\right)\left(x_{s(m),k}\right)\right) + \\ &\qquad\quad d_{X_k}\left(\left(e_{N,k} \circ p_{k,N}\right)\left(x_{s(m),k}\right), \left(e_{N,k} \circ p_{k,N}\right)\left(x_{s(n),k}\right)\right) + \\ &\qquad\quad d_{X_k}\left(\left(e_{N,k} \circ p_{k,N}\right)\left(x_{s(n),k}\right), x_{s(n),k}\right) \\ &\quad \leq \quad \delta\left(e_{k,N}, p_{N,k}\right) + \\ &\qquad\quad d_{X_N}\left(p_{k,N}\left(x_{s(m),k}\right), p_{k,N}\left(x_{s(n),k}\right)\right) + \\ &\qquad\quad \delta\left(e_{k,N}, p_{N,k}\right) \quad [\text{definition of } \delta,\ e_{N,k} \text{ is nonexpansive}] \\ &\quad \leq \quad \tfrac{\gamma}{3} + d_{X_N}\left(x_{s(m),N}, x_{s(n),N}\right) + \tfrac{\gamma}{3} \quad [(4.13)] \\ &\quad \leq \quad \tfrac{\gamma}{3} + \tfrac{\gamma}{3} + \tfrac{\gamma}{3} \quad [(4.14)] \end{aligned}$$

Consequently, we can conclude that, for all m and n,

$$\begin{aligned} &d_X\left(\bar{x}_{s(m)}, \bar{x}_{s(n)}\right) \\ &\quad = \quad d_X\left((x_{s(m),k})_k, (x_{s(n),k})_k\right) \\ &\quad = \quad \sup\left\{ d_{X_k}\left(x_{s(m),k}, x_{s(n),k}\right) \mid k \in \mathbb{N} \right\} \\ &\quad = \quad \sup\left\{ d_{X_k}\left(x_{s(m),k}, x_{s(n),k}\right) \mid k > N \right\} \quad [(4.12)] \\ &\quad \leq \quad \gamma \quad [(4.15)] \end{aligned}$$

Third, the fact that the spaces (X, d_X) and $F(X, d_X)$ are isomorphic can be proved along the lines of the proof of Theorem 5.23 of [RT92].

$\square$

By means of the above theorem we will prove Theorem 4.6. Before we come to this proof we make a remark on the above theorem.

In Lemma 3.10 of [AR89], it is shown that the direct limit of an ω-chain in CMS^E is an object in CMS^E. In the above proof, it is shown that the direct limit of a Cauchy ω-chain in KMS^E is an object in KMS^E. The fact that the ω-chain is Cauchy is essential can be demonstrated as follows. Consider the functor (4.11). Let $((X_n, d_{X_n}), (e_n, p_n)_n)$ be an ω-chain constructed from the functor as described above. This ω-chain is not Cauchy and one can easily verify that its direct limit is not in KMS^E.

The reader may be surprised by this, since in general topology the limit of an inverse sequence of nonempty and compact metrizable spaces is a nonempty and compact metrizable space (see the Theorems 3.2.13 and 4.2.5 of [Eng89]). However, in the topological setting continuous rather than nonexpansive functions are considered.

We conclude this subsection with the already announced

Proof of Theorem 4.6 Let $F : KMS \to KMS$ be a locally contractive functor. According to Proposition 4.12, $F^E : KMS^E \to KMS^E$ is a contractive functor. From Theorem 4.13 we can conclude that F^E has a fixed point. One can easily verify that this fixed point is also a fixed point of F. $\square$

4.2 Uniqueness of fixed points

The fixed points of locally contractive functors on KMS and contractive and locally contractive functors on KMS^E are shown to be unique up to isomorphism, that is, there exists a fixed point (as we have already seen in the Theorems 4.6 and 4.13) and all fixed points are isomorphic (as is shown in the following theorem).

Theorem 4.14 *Let $\mathcal{C}$ be a CMS-category. Let $F : \mathcal{C} \to \mathcal{C}$ be a locally contractive functor. If X and Y are fixed points of F then they are isomorphic.*

Proof Let $F : \mathcal{C} \to \mathcal{C}$ be a locally ϵ-Lipschitz functor, with $0 \le \epsilon < 1$. Let X and Y be fixed points of F. Then there exist arrows $f : X \to F(X)$, $g : F(X) \to X$, $h : Y \to F(Y)$, and $i : F(Y) \to Y$ satisfying

$$\begin{aligned} g \circ f &= id_X \\ f \circ g &= id_{F(X)} \\ i \circ h &= id_Y \\ h \circ i &= id_{F(Y)} \end{aligned}$$

We will define an arrow from X to Y and arrow from Y to X, and prove that they form an isomorphism.

In order to define a suitable arrow from X to Y we introduce the function

$$\Phi : hom(X, Y) \to hom(X, Y)$$

defined by

$$\Phi(j) = i \circ F(j) \circ f.$$

This function is a contraction, because for all j, $k \in hom(X, Y)$,

$$\begin{aligned} & d(\Phi(j), \Phi(k)) \\ = \quad & d(i \circ F(j) \circ f, i \circ F(k) \circ f) \\ \le \quad & d(F(j), F(k)) \quad [\circ \text{ is nonexpansive}] \\ \le \quad & \epsilon \cdot d(j, k) \quad [F \text{ is locally } \epsilon\text{-Lipschitz}] \end{aligned}$$

According to Banach's theorem (Theorem 1.4), Φ has a unique fixed point $\mathit{fix}(\Phi)$: an arrow from X to Y. Similarly, the unique fixed point of the function

$$\Psi : \mathit{hom}(Y, X) \to \mathit{hom}(Y, X)$$

defined by

$$\Psi(j) = g \circ F(j) \circ h,$$

which we denote by $\mathit{fix}(\Psi)$, is an arrow from Y to X.

We prove

$$\mathit{fix}(\Psi) \circ \mathit{fix}(\Phi) = \mathit{id}_X,$$

by uniqueness of fixed point. We introduce the function

$$\Omega : \mathit{hom}(X, X) \to \mathit{hom}(X, X)$$

defined by

$$\Omega(j) = g \circ F(j) \circ f$$

One can easily verify that Ω is contractive. Next, we will show that both $\mathit{fix}(\Psi) \circ \mathit{fix}(\Phi)$ and id_X are fixed point of Ω. We have that

$$\begin{aligned} & \Omega(\mathit{fix}(\Psi) \circ \mathit{fix}(\Phi)) \\ = \quad & g \circ F(\mathit{fix}(\Psi) \circ \mathit{fix}(\Phi)) \circ f \\ = \quad & g \circ F(\mathit{fix}(\Psi)) \circ F(\mathit{fix}(\Phi)) \circ f \quad [F \text{ preserves compositions}] \\ = \quad & g \circ F(\mathit{fix}(\Psi)) \circ h \circ i \circ F(\mathit{fix}(\Phi)) \circ f \\ = \quad & \Psi(\mathit{fix}(\Psi)) \circ \Phi(\mathit{fix}(\Phi)) \\ = \quad & \mathit{fix}(\Psi) \circ \mathit{fix}(\Phi) \quad [\text{fixed point property of } \Phi \text{ and } \Psi] \end{aligned}$$

and

$$\begin{aligned} & \Omega(\mathit{id}_X) \\ = \quad & g \circ F(\mathit{id}_X) \circ f \\ = \quad & g \circ \mathit{id}_{F(X)} \circ f \quad [F \text{ preserves identities}] \\ = \quad & \mathit{id}_X. \end{aligned}$$

Similarly, by means of the contractive function

$$\Upsilon : \mathit{hom}(Y, Y) \to \mathit{hom}(Y, Y)$$

defined by

$$\Upsilon(j) = i \circ F(j) \circ h$$

we can prove

$$\mathit{fix}(\Phi) \circ \mathit{fix}(\Psi) = \mathit{id}_Y$$

by uniqueness of fixed point. □

THEOREM 4.15 *Let $F : KMS \to KMS$ be a functor. If F is locally contractive then it has a unique fixed point.*

PROOF Immediate consequence of the Theorems 4.6 and 4.14. □

THEOREM 4.16 *Let $F : KMS^E \to KMS^E$ be a functor. If F is contractive and locally contractive then it has a unique fixed point.*

PROOF Immediate consequence of the Theorems 4.13 and 4.14. □

4.3 Lipschitz coefficients

In the above, we have seen that locally contractive functors on KMS and contractive functors on KMS^E have fixed points. In this subsection, for functors built from some basic functors by composition and tupling we will present a simple method for determining whether they are locally contractive and contractive.

For these functors we will approximate its so-called *locally Lipschitz coefficient* and *Lipschitz coefficient* in a compositional way: first, we will determine the (locally) Lipschitz coefficients for the basic functors, and second, we will show how the (locally) Lipschitz coefficient of a composition or tupling of functors can be approximated by the (locally) Lipschitz coefficients of the functors. The (locally) Lipschitz coefficient of a functor is defined such that the functor is (locally) contractive if and only if its (locally) Lipschitz coefficient is smaller than 1.

The locally Lipschitz coefficient of a functor is introduced in

Definition 4.17 Let $\mathcal{C}$ and $\mathcal{D}$ be CMS-categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a locally Lipschitz functor. The *locally Lipschitz coefficient* of F is defined by

$$LLC\,(F) = \min\{\,\epsilon \mid F \text{ is locally } \epsilon\text{-Lipschitz}\,\}.$$

One can easily verify that the minimum in the above definition exists. The basic functors on KMS and KMS^E have the following locally Lipschitz coefficients.

Proposition 4.18

$$\begin{array}{ll}
LLC\,(\epsilon\cdot) & = \epsilon \\
LLC\,(\mathcal{P}_{nk}) & = 1 \\
LLC\,(X, d_X) & = 0 \\
LLC\,(\times) & = 1 \\
LLC\,(+) & = 1 \\
LLC\,(\to^1) & \leq 2 \\
LLC\,(\epsilon\cdot^E) & = \epsilon \\
LLC\,({\mathcal{P}_{nk}}^E) & = 1 \\
LLC\,((X, d_X)^E) & = 0 \\
LLC\,(\times^E) & = 1 \\
LLC\,(+^E) & = 1 \\
LLC\,({\to^1}^E) & \leq 2
\end{array}$$

Proof Easy and left to the reader. □

Note that for the functors $\to^1$ and ${\to^1}^E$ we have only given an approximation. For composition and tupling we have the following

Proposition 4.19 *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be CMS-categories. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, and $H : \mathcal{C} \to \mathcal{E}$ be locally Lipschitz functors.*

$$\begin{array}{ll}
LLC\,(G \circ F) & \leq LLC\,(G) \cdot LLC\,(F) \\
LLC\,(F, H) & \leq \max\{LLC\,(F), LLC\,(H)\}
\end{array}$$

Proof We only treat the first case. It suffices to prove that, if F is locally ϵ_F-Lipschitz and G is locally ϵ_G-Lipschitz, then $G \circ F$ is locally $\epsilon_G \cdot \epsilon_F$-Lipschitz. For all X, Y in $\mathcal{C}$ and f, $g \in hom\,(X, Y)$, we have that

$$\begin{array}{ll}
\multicolumn{2}{l}{d\,((G \circ F)\,(f), (G \circ F)\,(g))} \\
= & d\,(G\,(F\,(f)), G\,(F\,(g))) \\
\leq & \epsilon_G \cdot d\,(F\,(f), F\,(g)) \quad [G \text{ is } \epsilon_G\text{-Lipschitz}] \\
\leq & \epsilon_G \cdot \epsilon_F \cdot d\,(f, g) \quad [F \text{ is } \epsilon_F\text{-Lipschitz}]
\end{array}$$

□

Note that the $\leq$'s in the above proposition cannot be replaced by $=$'s.

Example 4.20 From the above propositions we can derive that the locally Lipschitz coefficient of the first functor introduced in Example 3.5 is

$$\begin{array}{ll} \multicolumn{2}{l}{LLC\,(\times \circ ((X, d_X), \tfrac{1}{2}\cdot))} \\ \leq & LLC\,(\times) \cdot LLC\,((X, d_X), \tfrac{1}{2}\cdot) \\ \leq & \max\{LLC\,(X, d_X), LLC\,(\tfrac{1}{2}\cdot)\} \\ = & \tfrac{1}{2}. \end{array}$$

Consequently, the functor is locally contractive. Also the second functor of Example 3.5 is locally contractive, since its locally Lipschitz coefficient is

$$\begin{array}{ll} \multicolumn{2}{l}{LLC\,(+ \circ ((Y, d_Y), \mathcal{P}_{nk} \circ \times \circ ((Z, d_Z), \tfrac{1}{2}\cdot)))} \\ \leq & LLC(+) \cdot LLC\,((Y, d_Y), \mathcal{P}_{nk} \circ \times \circ ((Z, d_Z), \tfrac{1}{2}\cdot)) \\ \leq & \max\{LLC\,(Y, d_Y), LLC\,(\mathcal{P}_{nk} \circ \times \circ ((Z, d_Z), \tfrac{1}{2}\cdot))\} \\ \leq & LLC\,(\mathcal{P}_{nk}) \cdot LLC\,(\times) \cdot LLC\,((Z, d_Z), \tfrac{1}{2}\cdot) \\ \leq & \max\{LLC\,(Z, d_Z), LLC\,(\tfrac{1}{2}\cdot)\} \\ = & \tfrac{1}{2}. \end{array}$$

For the functor of Example 4.10, we have that

$$\begin{array}{ll} \multicolumn{2}{l}{LLC\,(+^E \circ ((Y, d_Y)^E, \tfrac{1}{3}\cdot^E \circ \rightarrow^{1^E} \circ (1\cdot^E, 1\cdot^E)))} \\ \leq & LLC\,(+^E) \cdot LLC\,((Y, d_Y)^E, \tfrac{1}{3}\cdot^E \circ \rightarrow^{1^E} \circ (1\cdot^E, 1\cdot^E)) \\ \leq & \max\{LLC\,((Y, d_Y)^E), LLC\,(\tfrac{1}{3}\cdot^E \circ \rightarrow^{1^E} \circ (1\cdot^E, 1\cdot^E))\} \\ \leq & LLC\,(\tfrac{1}{3}\cdot^E) \cdot LLC\,(\rightarrow^{1^E}) \cdot LLC\,(1\cdot^E, 1\cdot^E) \\ \leq & \tfrac{2}{3} \cdot \max\{LLC(1\cdot^E), LLC(1\cdot^E)\} \\ = & \tfrac{2}{3}. \end{array}$$

Hence, the functor is locally Lipschitz.

The Lipschitz coefficient of a functor is defined as follows.

Definition 4.21 Let $\mathcal{C}$ be a *CMS*-category. Let $F : (\mathcal{C}^E)^m \rightarrow (\mathcal{C}^E)^n$ be a Lipschitz functor. The *Lipschitz coefficient* of F is defined by

$$LC\,(F) = \min\{\,\epsilon \mid F \text{ is } \epsilon\text{-Lipschitz}\,\}.$$

For the basic functors we have the following Lipschitz coefficients.

Proposition 4.22

$$\begin{array}{ll} LC\,(\epsilon\cdot^E) & = \epsilon \\ LC\,(\mathcal{P}_{nk}{}^E) & = 1 \\ LC\,((X, d_X)^E) & = 0 \\ LC\,(\times^E) & = 1 \\ LC\,(+^E) & = 1 \\ LC\,(\rightarrow^{1^E}) & \leq 2 \end{array}$$

PROOF Easy and left to the reader. □

Again, for $\to^{1^E}$ we only have an approximation. Composition and tupling are dealt with in

PROPOSITION 4.23 *Let $\mathcal{C}$ be a CMS-category. Let $F : (\mathcal{C}^E)^m \to (\mathcal{C}^E)^n$, $G : (\mathcal{C}^E)^n \to (\mathcal{C}^E)^k$, and $H : (\mathcal{C}^E)^m \to (\mathcal{C}^E)^k$ be Lipschitz functors.*

$$\begin{aligned} &LC\,(G \circ F) \leq LC\,(G) \cdot LC\,(F) \\ &LC\,(F, H) \quad \leq \max\{LC\,(F), LC\,(H)\} \end{aligned}$$

PROOF Similar to the proof of Proposition 4.19. □

EXAMPLE 4.24 From the above propositions we cannot conclude that the functor

$$\tfrac{3}{5}\cdot^E \circ \to^{1^E} \circ\, (1\cdot^E, \tfrac{1}{2}\cdot^E)$$

is contractive, since

$$\begin{aligned} &LC\,(\tfrac{3}{5}\cdot^E \circ \to^{1^E} \circ\, (1\cdot^E, \tfrac{1}{2}\cdot^E)) \\ &\quad \leq \quad LC\,(\tfrac{3}{5}\cdot^E) \cdot LC\,(\to^{1^E}) \cdot LC\,(1\cdot^E, \tfrac{1}{2}\cdot^E) \\ &\quad \leq \quad \tfrac{6}{5} \cdot \max\{LC\,(1\cdot^E), LC\,(\tfrac{1}{2}\cdot^E)\} \\ &\quad = \quad \tfrac{6}{5}. \end{aligned}$$

According to the Propositions 4.18 and 4.19 for Lipschitz and locally Lipschitz functors F, $G : KMS^E \to KMS^E$ we have that

$$\begin{aligned} &LLC\,(\to^{1^E} \circ\, (F, G)) \leq 2 \cdot \max\{LLC\,(F), LLC\,(G)\} \\ &LC\,(\to^{1^E} \circ\, (F, G)) \quad \leq 2 \cdot \max\{LC\,(F), LC\,(G)\} \end{aligned}$$

These approximations can be improved as follows.

PROPOSITION 4.25 *Let F, $G : KMS^E \to KMS^E$ be Lipschitz and locally Lipschitz functors.*

$$\begin{aligned} &LLC\,(\to^{1^E} \circ\, (F, G)) \leq LLC\,(F) + LLC\,(G) \\ &LC\,(\to^{1^E} \circ\, (F, G)) \quad \leq LC\,(F) + LC\,(G) \end{aligned}$$

PROOF Easy and left to the reader. □

EXAMPLE 4.26 By means of the above proposition, we can conclude that the functor of Example 4.24 is contractive, since

$$\begin{aligned} &LC\,(\tfrac{3}{5}\cdot^E \circ \to^{1^E} \circ\, (1\cdot^E, \tfrac{1}{2}\cdot^E)) \\ &\quad \leq \quad LC\,(\tfrac{3}{5}\cdot^E) \cdot LC\,(\to^{1^E} \circ\, (1\cdot^E, \tfrac{1}{2}\cdot^E)) \\ &\quad \leq \quad \tfrac{3}{5} \cdot (LC\,(1\cdot^E) + LC\,(\tfrac{1}{2}\cdot^E)) \\ &\quad = \quad \tfrac{9}{10}. \end{aligned}$$

A more involved functor - the domain defined by a closely related functor is used in [BB93] - is discussed in

EXAMPLE 4.27 Consider the functor $F : KMS^E \to KMS^E$ defined by

$$F = \tfrac{1}{2}\cdot^E \circ +^E \circ\, ((U, d_U)^E, G),$$

where

$$G = \rightarrow^{1\,E} \circ (H, \mathcal{P}_{nk}{}^{E} \circ \times^{E} \circ (\rightarrow^{1\,E} \circ (H, \tfrac{1}{2}\cdot^{E}), \tfrac{1}{2}\cdot^{E}))$$

and

$$H = \tfrac{1}{2}\cdot^{E} \circ \rightarrow^{1\,E} \circ ((V, d_V)^E, \tfrac{1}{2}\cdot^{E})$$

with (U, d_U) and (V, d_V) 1-bounded compact metric spaces. The functor corresponds to the domain equation

$$(X, d_X) \cong \tfrac{1}{2} \cdot ((U, d_U) + (Y, d_Y))$$

where

$$(Y, d_Y) \cong (Z, d_Z) \rightarrow^1 \mathcal{P}_{nk}(((Z, d_Z) \rightarrow^1 \tfrac{1}{2} \cdot (X, d_X)) \times \tfrac{1}{2} \cdot (X, d_X))$$

and

$$(Z, d_Z) \cong \tfrac{1}{2} \cdot ((V, d_V) \rightarrow^1 \tfrac{1}{2} \cdot (X, d_X)).$$

We have that

$$\begin{array}{rcl}
LLC\,(H) & \leq & LLC\,(\tfrac{1}{2}\cdot^{E}) \cdot LLC\,(\rightarrow^{1\,E} \circ ((V, d_V)^E, \tfrac{1}{2}\cdot^{E})) \\
 & \leq & \tfrac{1}{2} \cdot (LLC\,((V, d_V)^E) + LLC\,(\tfrac{1}{2}\cdot^{E})) \\
 & = & \tfrac{1}{4} \\
LLC\,(G) & \leq & LLC\,(H) + LLC\,(\mathcal{P}_{nk}{}^{E} \circ \times^{E} \circ (\rightarrow^{1\,E} \circ (H, \tfrac{1}{2}\cdot^{E}), \tfrac{1}{2}\cdot^{E})) \\
 & \leq & \tfrac{1}{4} + LLC\,(\mathcal{P}_{nk}{}^{E}) \cdot LLC\,(\times^{E}) \cdot LLC\,(\rightarrow^{1\,E} \circ (H, \tfrac{1}{2}\cdot^{E}), \tfrac{1}{2}\cdot^{E})) \\
 & \leq & \tfrac{1}{4} + \max\{LLC\,(\rightarrow^{1\,E} \circ (H, \tfrac{1}{2}\cdot^{E})), LLC\,(\tfrac{1}{2}\cdot^{E})\} \\
 & \leq & \tfrac{1}{4} + \max\{LLC\,(H) + LLC\,(\tfrac{1}{2}\cdot^{E}), \tfrac{1}{2}\} \\
 & \leq & 1 \\
LLC\,(F) & \leq & LLC\,(\tfrac{1}{2}\cdot^{E}) \cdot LLC\,(+^{E}) \cdot LLC\,((U, d_U)^E, G) \\
 & \leq & \tfrac{1}{2} \cdot \max\{LLC\,((U, d_U)^E), LLC\,(G)\} \\
 & = & \tfrac{1}{2}
\end{array}$$

Similarly, one can demonstrate that

$$\begin{array}{l}
LC\,(H) \leq \tfrac{1}{4} \\
LC\,(G) \leq 1 \\
LC\,(F) \leq \tfrac{1}{2}
\end{array}$$

Consequently, the functor F is contractive and locally contractive. Hence, F has a unique fixed point.

This concludes our discussion of fixed points of functors, and brings us to the end of this report.

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