A Note on the Hutchinson Metric

(Draft)

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July 20, 1999

In [Hut81], John Hutchinson introduced a metric on the set of Borel probability measures on a metric space. In this note, I present the Hutchinson metric and prove several properties of this metric. I restrict my attention to 1-bounded metric spaces. Most of the material in this note is based on [Par67] and [Edg98]. Proposition 7, 8 and 9 may be new.

Let X be a 1-bounded metric space. I denote the set of Borel probability measures on X by $\mathcal{M}(X)$. The Hutchinson metric on $\mathcal{M}(X)$ is introduced in

Definition 1 The function $d_{\mathcal{M}(X)} : \mathcal{M}(X) \times \mathcal{M}(X) \to [0,1]$ is defined by

$$d_{\mathcal{M}(X)}(\mu,\nu) = \sup \left\{ \left| \int_X f d\mu - \int_X f d\nu \right| : f \in X \to [0,1] \text{ is nonexpansive} \right\}.$$

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First, I show that the above introduced distance function is a metric. My proof is based on [Edg98, Proposition 2.5.14].

Proposition 2 *$d_{\mathcal{M}(X)}$ is a metric.*

Proof Since for all nonexpansive $f \in X \to [0,1]$,

$$0 = \int_X 0 d\mu \leq \int_X f d\mu \leq \int_X 1 d\mu = 1,$$

$d_{\mathcal{M}(X)}(\mu,\nu) \leq 1$.
Obviously, $d_{\mathcal{M}(X)}(\mu,\mu) = 0$.
Towards a contradiction, assume that $\mu \neq \nu$ and $\int_X f d\mu = \int_X f d\nu$ for all nonexpansive $f \in X \to [0,1]$. Since Borel probability measures on X are completely determined by their restrictions to the closed subsets of X, $\mu(C) \neq \nu(C)$ for some closed subset C of X. For each $n \in \mathbb{N}$, the function $f_n : X \to [0,1]$ is defined by

$$f_n(x) = \max\{0, \tfrac{1}{n} - \inf_{c \in C} d_X(x,c)\}.$$

Since for all x, $y \in X$,

$$\begin{aligned} & |f_n(x) - f_n(y)| \\ &= |\max\{0, \tfrac{1}{n} - \inf_{c \in C} d_X(x,c)\} - \max\{0, \tfrac{1}{n} - \inf_{c \in C} d_X(y,c)\}| \\ &\leq |\tfrac{1}{n} - \inf_{c \in C} d_X(x,c) - (\tfrac{1}{n} - \inf_{c \in C} d_X(y,c))| \end{aligned}$$

$$\begin{aligned}
&= |\inf_{c\in C} d_X(x,c) - \inf_{c\in C} d_X(y,c)| \\
&\leq d_X(x,y) \quad \text{[see [Par67, Theorem I.1.1]]}
\end{aligned}$$

f_n is nonexpansive. One can easily verify that the set $\{ x \in X : \inf_{c\in C} d_X(x,c) < \frac{1}{n} \}$ is open and that

$$\tfrac{1}{n} \cdot \chi_C \leq f_n \leq \tfrac{1}{n} \cdot \chi_{\{ x\in X : \inf_{c\in C} d_X(x,c) < \frac{1}{n} \}}.$$

Therefore,

$$\begin{aligned}
&\mu(C) \\
&= \int_X \chi_C \, d\mu \\
&\leq n \cdot \int_X f_n \, d\mu \\
&= n \cdot \int_X f_n \, d\nu \\
&\leq \int_X \chi_{\{ x\in X : \inf_{c\in C} d_X(x,c) < \frac{1}{n} \}} \, d\nu \\
&= \nu(\{ x \in X : \inf_{c\in C} d_X(x,c) < \tfrac{1}{n} \})
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mu(C) \\
&\leq \inf_{n\in \mathbb{N}} \nu(\{ x \in X : \inf_{c\in C} d_X(x,c) < \tfrac{1}{n} \}) \\
&= \nu(\bigcap_{n\in\mathbb{N}} \{ x \in X \mid \inf_{c\in C} d_X(x,c) < \tfrac{1}{n} \}) \\
&= \nu(C).
\end{aligned}$$

Similarly, one can show that $\nu(C) \leq \mu(C)$ which leads to a contradiction.

Clearly, $d_{\mathcal{M}(X)}(\mu,\nu) = d_{\mathcal{M}(X)}(\nu,\mu)$.

For each nonexpansive $f \in X \to [0,1]$,

$$\begin{aligned}
&\left| \int_X f\,d\mu - \int_X f\,d\omega \right| \\
&\leq \left| \int_X f\,d\mu - \int_X f\,d\nu \right| + \left| \int_X f\,d\nu - \int_X f\,d\omega \right| \\
&\leq d_{\mathcal{M}(X)}(\mu,\nu) + d_{\mathcal{M}(X)}(\nu,\omega).
\end{aligned}$$

Hence,

$$d_{\mathcal{M}(X)}(\mu,\omega) \leq d_{\mathcal{M}(X)}(\mu,\nu) + d_{\mathcal{M}(X)}(\nu,\omega).$$

□

The metric space X can be isometrically embedded into the metric space $\mathcal{M}(X)$ (cf. [Par67, Lemma II.6.1] and [Edg98, page 108]). For each $x \in X$, let δ_x be the Dirac measure at x.

PROPOSITION 3 *δ is isometric.*

PROOF Let $x, y \in X$.

$$d_{\mathcal{M}(X)}(\delta_x, \delta_y)$$

$$
\begin{aligned}
&= \sup\left\{\left|\int_X f d\delta_x - \int_X f d\delta_y\right| : f \in X \to [0,1] \text{ is nonexpansive}\right\} \\
&= \sup\{\,|f(x) - f(y)| : f \in X \to [0,1] \text{ is nonexpansive}\} \\
&\leq d_X(x,y).
\end{aligned}
$$

The function $g : X \to [0,1]$ defined by

$$g(z) = d_X(x,z)$$

is nonexpansive and

$$
\begin{aligned}
&\left|\int_X g\, d\delta_x - \int_X g\, d\delta_y\right| \\
&= |g(x) - g(y)| \\
&= d_X(x,y).
\end{aligned}
$$

□

As a consequence, if $\mathcal{M}(X)$ is complete then so is X.

I denote the set of tight Borel probability measures on X by $\mathcal{M}_t(X)$. Since the Dirac measures are tight, the metric space X can be isometrically embedded into the metric space $\mathcal{M}_t(X)$. As a consequence, if $\mathcal{M}_t(X)$ is complete then so is X.

PROPOSITION 4 *If X is complete then $\mathcal{M}_t(X)$ is complete.*

PROOF I refer the reader to [Edg98, Theorem 2.5.25] for the moment. □

$\mathcal{M}_t$ can be extended to an endofunctor on the category *CMS* of 1-bounded complete metric spaces and nonexpansive functions as follows.

DEFINITION 5 Let X and Y be 1-bounded complete metric spaces. Let $f : X \to Y$ be nonexpansive. The function $\mathcal{M}_t(f) : \mathcal{M}_t(X) \to \mathcal{M}_t(Y)$ is defined by

$$\mathcal{M}_t(f)(\mu) = \mu \circ f^{-1}.$$

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PROPOSITION 6 *Let $f : X \to Y$ be nonexpansive. If $\mu \in \mathcal{M}_t(X)$ then $\mu \circ f^{-1} \in \mathcal{M}_t(Y)$.*

PROOF Let $\epsilon > 0$. Since μ is tight, there exists a compact subset K_ϵ of X such that $\mu(X \setminus K_\epsilon) < \epsilon$. Because f is nonexpansive, $f(K_\epsilon)$ is a compact subset of Y. Since $f^{-1}(Y \setminus f(K_\epsilon))$ is a subset of $X \setminus K_\epsilon$, $(\mu \circ f^{-1})(Y \setminus f(K_\epsilon)) < \epsilon$. Hence, $\mu \circ f^{-1}$ is tight. □

PROPOSITION 7 *$\mathcal{M}_t(f)$ is nonexpansive.*

PROOF For all μ, $\nu \in \mathcal{M}_t(X)$,

$$
\begin{aligned}
&d_{\mathcal{M}_t(Y)}(\mathcal{M}_t(f)(\mu), \mathcal{M}_t(f)(\nu)) \\
&= \sup\left\{\left|\int_Y g\, d(\mu \circ f^{-1}) - \int_Y g\, d(\nu \circ f^{-1})\right| : g \in Y \to [0,1] \text{ is nonexpansive}\right\} \\
&= \sup\left\{\left|\int_X (g \circ f)\, d\mu - \int_X (g \circ f)\, d\nu\right| : g \in Y \to [0,1] \text{ is nonexpansive}\right\} \\
&\leq \sup\left\{\left|\int_X h\, d\mu - \int_X h\, d\nu\right| : h \in X \to [0,1] \text{ is nonexpansive}\right\} \\
&= d_{\mathcal{M}_t(X)}(\mu, \nu).
\end{aligned}
$$

□

This functor is locally nonexpansive (see [RT92, Definition 4.2]).

PROPOSITION 8 *$\mathcal{M}_t$ is locally nonexpansive.*

PROOF For all nonexpansive f, $g \in X \to Y$ and $\mu \in \mathcal{M}_t(X)$,

$$\begin{array}{ll} \multicolumn{2}{l}{d_{\mathcal{M}_t(Y)}(\mathcal{M}_t(f)(\mu), \mathcal{M}_t(g)(\mu))} \\ = & \sup \left\{ \left| \int_Y h\, d(\mu \circ f^{-1}) - \int_Y h\, d(\mu \circ g^{-1}) \right| : h \in Y \to [0,1] \text{ is nonexpansive} \right\} \\ = & \sup \left\{ \left| \int_X (h \circ f)\, d\mu - \int_X (h \circ g)\, d\mu \right| : h \in Y \to [0,1] \text{ is nonexpansive} \right\} \\ = & \sup \left\{ \left| \int_X (h \circ f - h \circ g)\, d\mu \right| : h \in Y \to [0,1] \text{ is nonexpansive} \right\} \\ \leq & d_{X \to Y}(f, g), \end{array}$$

since for all nonexpansive $h \in Y \to [0,1]$ and $x \in X$,

$$\begin{array}{ll} \multicolumn{2}{l}{(h \circ f - h \circ g)(x)} \\ \leq & |(h \circ f)(x) - (h \circ g)(x)| \\ \leq & d_Y(f(x), g(x)) \quad [h \text{ is nonexpansive}] \\ \leq & d_{X \to Y}(f, g). \end{array}$$

□

The terminal object and the coproduct of the category *CMS* are denoted by 1 and $+$, respectively. Let $0 < \epsilon < 1$. The operation $\epsilon\cdot$ on metric spaces leaves the set unchanged and multiplies the metric by ϵ. This operation can be extended straightforwardly to an endofunctor on *CMS*.

PROPOSITION 9 *For all $0 < \epsilon < 1$, there exists a terminal $1 + \epsilon \cdot \mathcal{M}_t(-)$-coalgebra.*

PROOF Let $0 < \epsilon < 1$. Since the functor $\mathcal{M}_t$ is locally nonexpansive, the functor $1 + \epsilon \cdot \mathcal{M}_t(-)$ is locally contractive. According to [RT92, Theorem 4.8], there exists a terminal $1 + \epsilon \cdot \mathcal{M}_t(-)$-coalgebra. □

References

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