From Branching to Linear Metric Domains
(and back)*

Franck van Breugel†
McGill University
School of Computer Science
3480 University Street, Montreal H3A 2A7, Canada

Abstract

Besides partial orders, also *metric spaces* have turned out to be very useful to give semantics to programming languages (see, e.g., the collection of papers of the Amsterdam Concurrency Group [BR92]). In the literature, one encounters two main classes of metric domains: *linear domains* and *branching domains*. Linear domains were already studied by topologists in the early twenties. Branching domains have been introduced by, e.g., De Bakker and Zucker [BZ82, BZ83], Golson and Rounds [GR83, Gol84], and the author [Bre93]. The elements of these linear and branching domains are convenient to model—one might even say that they represent—*trace equivalence* classes and *bisimulation equivalence* classes, respectively. The former is a simple observation. The latter has been proved by Van Glabbeek and Rutten [GR89].

Linear domains are more abstract than branching domains. Our aim is to show that linear domains can be embedded in branching domains. We focus on the branching domain $\mathcal{B}$ introduced by De Bakker and Zucker in [BZ83] and the linear domain $\mathcal{L}$ the elements of which can be viewed as nonempty and compact sets of sequences.

By abstracting from the branching structure of the branching domain $\mathcal{B}$ we arrive at the linear domain $\mathcal{L}$. This abstraction operator—called *linearize operator* in the sequel—can be defined conveniently in terms of a *metric labelled transition system*. The theory of metric labelled transition systems has been outlined in the author's [Bre94a] and has been developed further in his thesis [Bre94b]. The branching domain $\mathcal{B}$ can be seen as a labelled transition system as De Bakker,

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Bergstra, Klop, and Meyer noted in [BBKM84]. It can even be viewed as a metric labelled transition system. This metric labelled transition system is *compactly branching*—being a generalization of finitely branching. As a consequence, we can apply a theorem—being a generalization of a theorem reminiscent to König's lemma—obtaining the linearize operator *lin*. The additional metric structure of a metric labelled transition system (with respect to a labelled transition systems) is essential in the above. Similarly, we can linearize the more involved branching domains—used to model object-oriented and higher-order features—introduced by Rutten [Rut90] and De Bakker and the author [BB93].

Also the linear domain $\mathcal{L}$ can be viewed as a compactly branching metric labelled transition system. This can be done in two obvious ways. Both compactly branching metric labelled transition systems give rise to a *branch operator*: $branch_0$ and $branch_1$.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, shift left=2, "branch_0"] \arrow[r, shift right=2, "branch_1"'] & \mathcal{B} \arrow[l, "lin"']
\end{tikzcd}$$

The linearize and branch operators are related as follows:

$$lin \circ branch_0 = id_{\mathcal{L}} \text{ and } lin \circ branch_1 = id_{\mathcal{L}}.$$

We make the relationship between the linearize and branch operators even more precise. We follow the work of Nielsen and Winskel et al. [Win84, SNW93, WN94] using category theory—in particular functors—to classify the domains. The linear and branching domains are both turned into a *quasimetric space* which induces a preorder and hence a category. Lately, there is a growing interest in quasimetric spaces. See, e.g., Wagner's thesis [Wag94] and Flagg and Kopperman's [FK94]. The quasimetrics are obtained from the metrics the d omains are endowed with by dropping one half of the Hausdorff metric. The morphisms of the branching domain can be seen as simulations and the morphisms of the linear domains can be viewed simply as inclusion functions. The linearize operator and both the branch operators are functors. These functors form a reflection and a coreflection—our main result.

$$\begin{tikzcd}
\mathcal{L} \arrow[r, hook, shift left=2] \arrow[r, hook, shift right=2, -<] & \mathcal{B}
\end{tikzcd}$$

By means of this reflection and coreflection we have expressed that the linear domain $\mathcal{L}$ can be embedded in the branching domain $\mathcal{B}$. These adjunctions can also be used for the transfer of (categorical) techniques from one domain to the other.

The details of the above will appear in [Bre95].

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