York University COSC6115 March 21, 2001

Homework Assignment #2
Due: April 9, 2001

1. [5 marks] Let $\mathbf{E} = \bigcup_{c>0} \mathbf{TIME}(c^n)$. Is it true that $\mathbf{E} = \mathbf{TIME}(2^n)$? Justify your answer.

2. [5 marks] We say that a language L is *polytime-verifiable* if there exists a constant c and a polynomial-time algorithm A that takes two string arguments x and y and outputs either yes or no, such that

$$L = \{x \in \Sigma^* : \text{ there exists a string } y \text{ with } |y| \leq |x|^c \text{ such that } A(x, y) \text{ outputs yes}\}.$$

Show that a language is in **NP** if and only if it is polytime-verifiable.

3. [10 marks]

[7] **(a)** Show that **NL** is closed under reductions.

[3] **(b)** Let $\mathcal{C}$ be a class that is closed under reductions. Show that co-$\mathcal{C}$ is also closed under reductions.

4. [15 marks] If x is a string, let $R(x)$ denote the reversal of x. (For example, $R(001011)$ is 110100.) Let $x \cdot y$ denote the concatenation of strings x and y. Let PALINDROMES be the language $\{x \in \{0,1\}^* : x = R(x)\}$. We saw in class how to decide this language in $O(n)$ time using a 2-tape TM and in $O(n^2)$ time using a single-tape TM. Here we show that any single-tape TM that decides L requires $\Omega(n^2)$ time.

Let M be any single-tape TM that decides PALINDROMES. Let q be the number of states in M's state set. Let x be an input string, and i be a natural number. During M's computation on input x, the cursor will cross the boundary between the ith tape square and the $(i+1)$th square (in either direction) some number of times (possibly zero). Let $S(x, i)$ be the a list whose jth element is the state of M just after the cursor crosses the boundary for the jth time. (We number the tape squares so that the 0th square contains $\triangleright$ and the input string is initially written starting in the first square.)

[2] **(a)** Explain briefly why the running time of M on input x is at least $\sum_{i=0}^{\infty} |S(x, i)|$, where $|S(x, i)|$ denotes the length of the list $S(x, i)$. Can the running time ever be strictly greater than this sum?

[4] **(b)** Let a, b, c and d be strings. Suppose that $a \cdot b$ and $c \cdot d$ are in PALINDROMES, and that $S(a \cdot b, |a|) = S(c \cdot d, |c|)$. Show that M must accept $a \cdot d$.

[2] **(c)** Fix an input length n that is a multiple of 3. We call a list of states "short" if its length is less than $n/(6 \log_2 q)$, and we call it "long" otherwise. Show that there are fewer than $2^{n/6}$ possible short lists.

[3] **(d)** Show that, for each i satisfying $n/3 \le i \le 2n/3$, there are at most $2^{n/6}$ different binary strings y of length $n/3$ such that $S(y \cdot 0^{n/3} \cdot R(y), i)$ is short. (Here, $0^{n/3}$ denotes the string consisting of n/3 0's.)

[4] **(e)** Show that, when n is a large multiple of 3, there is an input of length n for which M executes at least $n^2/(18 \log_2 q)$ steps.

5. [15 marks] Problem 2.8.17 in the textbook. You can assume that $f(n)$ is a proper function with $f(n) \ge n$. Hint: notice that a computation of a k-tape non-deterministic TM can be described by a sequence of tuples, each tuple being an element of the transition relation Δ. Given a sequence of random tuples, how can you check (efficiently!) whether it is a legal computation using only two tapes?