Efficient Algorithm for Approximating Maximum Inscribed Sphere in High Dimensional Polytope

[Extended Abstract]

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ABSTRACT

In this paper, we consider the problem of computing a maximum inscribed sphere inside a high dimensional polytope formed by a set of halfspaces (or linear constraints) and with bounded aspect ratio, and present an efficient algorithm for computing a $(1-\epsilon)$-approximation of the sphere. More specifically, given any aspect-ratio-bounded polytope P defined by n d-dimensional halfspaces, an interior point O of P, and a constant $\epsilon > 0$, our algorithm computes in $O(nd/\epsilon^3)$ time a sphere inside P with a radius no less than $(1-\epsilon)R_{opt}$, where R_{opt} is the radius of a maximum inscribed sphere of P. Our algorithm is based on the core-set concept and a number of interesting geometric observations. Our result solves a special case of an open problem posted by Khachiyan and Todd [13].

Categories and Subject Descriptors

F.2.2 [**Nonnumerical Algorithms and Problems**]: Computations on discrete structures; Geometrical problems and computations

General Terms

Algorithms,Theory

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Keywords

Maximum Inscribed Sphere, Approximation, Polytope, Core-Set, Chebyshev Center

1. INTRODUCTION

In this paper, we consider the following problem of approximating maximum inscribed sphere (MaxIS): Given a set $\mathcal{H} = \{H_1, H_2, \cdots, H_n\}$ of halfspace in d-dimensional Euclidean space E^d, a point O in the common intersection P of $\mathcal{H}$, and a small constant $\epsilon > 0$, compute a sphere B inside P (assume that P is bounded) with a radius no less than $(1-\epsilon)R_{opt}$, where R_{opt} is the radius of a maximum inscribed sphere of P whose center is also called a Chebyshev center of P.

Efficiently computing the maximum inscribed sphere or ellipsoid in high dimensional polytope formed by a set of halfspaces (or linear constraints) is a challenging problem in theoretical computer science and operations research. It is closely related to the interior-point method for linear programming (LP) [17]. One commonly used approach [6] for solving the MaxIS problem is to reduce it to a linear programming (LP) problem with one more dimension and then use existing LP algorithms to solve it. Thus optimally solving the MaxIS problem could be rather costly (i.e., with the same complexity as an LP problem). With the power of core-sets [1, 5], it has been shown that many problems (such as the clustering problems and shape fitting problems [1, 10, 9]) in high dimensional space which originally have high time complexities can now be efficiently approximated. For instance, using core-sets, a $(1+\epsilon)$-approximation of the minimum enclosing sphere (MinES) of a set of n points in d-dimensional space can be computed in linear time [1, 5, 4, 16, 12]. Thus it would be very interesting to know whether the core-set concept can be used to speed up the computation of the MaxIS problem. Note that when the dimensionality is a constant, the ϵ-kernel result in [1] implies a linear time approximation algorithm for the MaxIS problem.

A problem closely related to the MaxIS is that of computing the maximum inscribed ellipsoid (MaxIE) (also called maximum volume ellipsoid (MaxVE)). The MaxIE of a polytope is, in general, quite different from its MaxIS, but in some special cases, such as the traveling salesman polytope [3], the two objects are actually the same. The MaxIE has direct applications in the interior-point method and is frequently used to round convex bodies in R^d space. Due to its importance, the MaxIE has received a great deal of attention. Khachiyan and Todd [13] built a polynomial bound for computing an approximate point of the center of the maximal inscribed ellipsoid in the polytope defined by linear constraints. Most recently, Anstreicher [2] showed that a $(1-\epsilon)$-approximation of the maximum inscribed ellipsoid of the polytope can be computed in $O(n^{3.5}\log(\frac{nR}{\epsilon}))$ time, where R is a priori known ratio between the radii of a sphere enclosing P and its co-centered sphere inscribed in P. This is the best complexity to our knowledge. For the special case of $n \ll d^2$, Zhang and Gao [18] recently obtained a more practical algorithm with roughly the same time bound.

A dual problem of MaxIE is that of computing a minimal enclosing ellipsoid (MinEE) for the convex hull of n points in E^d. Khachiyan [11] showed that a $(1+\epsilon)$-approximation of the MinEE can be computed in $O(n^{3.5}\log(\frac{n}{\epsilon}))$ time. Recently, Kumar and Yildirim designed an efficient approximation algorithm by using core-sets and showed the existence of a core-set of size $\alpha = O(d(\log d + \frac{1}{\epsilon}))$. The complexity of their algorithm is $O(nd^2\alpha + \alpha^{4.5}\log(\frac{\alpha}{\epsilon}))$.

Khachiyan and Todd [13] observed that the MaxIE and MinEE problems are equivalent if they are appropriately represented, and conjectured that there exist *"linear-time"* reductions among the following 4 ellipsoids: $C:(1-\epsilon)$-approximation of the MinEE $\rightleftharpoons C_O:(1-\epsilon)$-approximation of the MinEE centered at an arbitrary point $O \rightleftharpoons I_O:(1-\epsilon)$-approximation of the MaxIE centered at an arbitrary interior point $O \Rightarrow I:(1-\epsilon)$-approximation of the MaxIE. They posted as an open problem for searching for the reduction from I_O to I.

In this paper we present an efficient algorithm for computing a $(1-\epsilon)$-approximation of the MaxIS of an aspect-ratio-bounded P. The running time of our algorithm is $O(\frac{nd}{\epsilon^3})$, which is linear in terms of the size of the input (i.e., nd). (The running time includes a constant factor of α^3, where α is the aspect ratio.) For simplicity of our analysis, we assume in our current version that $n \geq O(1/\epsilon^2)$ and ϵ is reasonably small, e.g., $0 < \epsilon \leq 0.1$. Our algorithm first moves the origin of the coordinate system to the interior point O and then applies the dual transform to reduce the MaxIS problem to a sequence of the MinES problems. Based on a number of interesting observations and the core-set concept, we prove that at each iteration, significant progress can be achieved. Particularly, we show that only $O(1/\epsilon^2)$ iterations are needed when aspect ratio is a constant, no matter where the interior point lies inside the polytope, and each iteration takes no more than $O(nd/\epsilon)$ time.

Our algorithm can be easily implemented and converges very quickly for randomly generated polytopes. It takes only a small constant number of steps to converge to the $(1-\epsilon)$-approximation of the MaxIS even for dimensions as high as 1000 and with very large aspect ratio. Detailed experimental results are left for the full paper. Our algorithm settles a special case of the open problem of Khachiyan and Todd [13].

2. FROM MAXIS TO MINES

To compute the $(1-\epsilon)$-approximation of the MaxIS of P, our main idea is to reduce the computation of the MaxIS problem to that of a sequence of MinES's. Without loss of generality, we assume that the origin O of the coordinate system is an interior point of P. Then we use the following dual transform to convert the bounding hyperplane of each halfspace H_i into a point for $1 \leq i \leq n$. For simplicity, we also use H_i to denote the hyperplane of the corresponding halfspace and H_i^* to denote its dual.

Let H be an arbitrary hyperplane $p_1x_1 + p_2x_2 + \cdots + p_dx_d = 1$ in E^d not containing the origin O. The dual H^* of H is the point $(p_1, p_2, \cdots, p_d) \in E^d$. The dual transform has several nice properties which can be summarized by the following lemma.

LEMMA 1. *If the dual space is superimposed on the primal space so that they share the same coordinate system, then the dual transform maps a hyperplane H at distance h from the origin O to a point H^* at distance $1/h$ from the origin and the ray $\vec{OH^*}$ is orthogonal to H.*

COROLLARY 1. *Let H_i and H_j be two hyperplanes in the primal space, and H_i^* and H_j^* be their corresponding dual points. If $dist(H_i,O) > dist(H_j,O)$, then in the dual space $dist(H_i^*,O) = \|H_i^*\| < dist(H_j,O) = \|H_j^*\|$.*

With the above dual transform, each halfspace H_i in $\mathcal{H}$ can be mapped to a point H_i^*. Let $\mathcal{H}^*$ be the set of dual points of $\mathcal{H}$. Since for the set of dual points, we can efficiently compute a good approximation of the minimum enclosing sphere, thus a natural question is "How do we make use of the MinES of $\mathcal{H}^*$ for the computation of the MaxIS of $\mathcal{H}$?" To answer this question, we first let the two spheres share the same space, that is, superimpose the dual space on the primal space so that they share the same coordinate system. (Hereafter, unless we specify otherwise, we always assume that the two spaces share the same coordinate system.) Then, we study how the two spheres change when the origin O moves around in P.

Let O and O' be the old and new origin and C be the center of the MinES B_{min} of $\mathcal{H}^*$. Let $\delta = \|CO\|$ and $s = \vec{OO'}$. We use r and r' to denote the radii of the MinES of $\mathcal{H}^*$ and $\mathcal{H}^{*'}$ respectively, where $\mathcal{H}^{*'}$ is the set of new dual points (defined by Lemma 2) after moving O to O'. For any point H^* in the dual space, we use $f(H^*)$ to denote the new dual point of the hyperplane corresponding to H^*. The following lemma shows how $f(H^*)$ changes when moving O to O'.

LEMMA 2. *Let H be an arbitrary hyperplane and H^* be its dual point. If the origin O is moved to a new location O' by a vector $s = \vec{OO'}$ and H remains at its original position (hence the distance between O and H changes), then after this movement, H^* moves accordingly to a new point $H^{*'}$*

$$H^{*'} = f(H^*) = \frac{H^*}{1 - H^* \cdot s},$$

where $H^* \cdot s$ is the inner product of H^* and s.

PROOF. Let the equation of H be $p_1x_1 + p_2x_2 + \cdots + p_dx_d = 1$. Then H^* is the point $(p_1, p_2, \cdots, p_d)$. Let s be $(s_1, s_2, \cdots, s_d)$. When O is moved to O', the new equation of H is

$$\frac{\sum_i p_i x_i}{1 - \sum_i p_i s_i} = 1,$$

which means $H^{*'}$ is at $\frac{H^*}{1-H^*\cdot s}$. $\square$

The following lemma shows an interesting relation between the centers of MaxIS and MinES.

LEMMA 3. *If the center C of the MinES of $\mathcal{H}^*$ coincides with the origin O, or say C and O overlap, then there exists a MaxIS of $\mathcal{H}$ co-centered with the MinES at O.*

To prove this lemma, we need the following lemmas.

LEMMA 4. *Let P be any bounded polytope defined by linear constraints, and O_{opt} be the center of a MaxIS of P. Let O be any interior point but not the center of any MaxIS of P, q be any point on the segment $O_{opt}O$, and $R(q)$ be the minimum distance from q to the boundary of P (i.e., $R(q)$ is the radius of the maximum inscribed sphere centered at q). Then $R(q)$ is monotonically decreasing when q moves from O_{opt} to O.*

PROOF. Suppose this is not true. Then there must exist two points q_1 and q_2, other than O_{opt}, on the segment $O_{opt}O$ with q_2 further away from O_{opt} than q_1 but $R(q_1) \leq R(q_2) < R(O_{opt})$. Let H be the closest hyperplane to q_1, and b_1, b_2, and b_3 be the closest points of q_1, q_2, and O_{opt} on H. Clearly, b_1, b_2, b_3, q_1, q_2 and O_{opt} are all on the same 2-D plane, say B, and b_1, b_2, b_3 are in the common intersection (i.e., a straight line l) of B and H. By the definition of $R()$, we known that

$$|q_2 b_2| \geq R(q_2) \geq R(q_1) = |q_1 b_1|.$$

Also since O_{opt} is the center of the MaxIS of P,

$$|O_{opt} b_3| \geq R(O_{opt}) > R(q_1) = |q_1 b_1|.$$

This means that b_1, b_2 and b_3 cannot be on the same straight line, a contradiction. □

The following lemma has been proved in [5].

LEMMA 5. *If $B(T)$ is the MinES of a set T of points in E^d, then any closed halfspace that contains the center $C_{B(T)}$ also contains a point of T that is at distance $r_{B(T)}$ from $C_{B(T)}$. It follows that for any point z at distance K from $C_{B(T)}$, there is a point $t \in T$ at distance at least $\sqrt{r^2_{B(T)} + K^2}$ from z.*

From this lemma, we know that the farthest point in $\mathcal{H}^*$ to O has a distance at least $\sqrt{r^2 + \delta^2}$ to O. Thus we have the following corollary.

COROLLARY 2. *The distance from O to the closest hyperplane $H \in \mathcal{H}$ is no more than $h = \frac{1}{\sqrt{r^2+\delta^2}}$.*

With the above two lemmas, we can now prove Lemma 3.

PROOF. Suppose C and O overlap but the maximum inscribed sphere centered at O is not the MaxIS of P. Then let O_{opt} be the closest point (to O) which is the center of a MaxIS of P. Consider the hyperplane V crossing O and orthogonal to the segment OO_{opt}. By Lemma 5, we know that on the side of V containing O_{opt}, there exists at least one point $H^* \in \mathcal{H}^*$ on the boundary of the MinES. When O moves towards O_{opt}, by Lemma 2 we know that the distance between H^* and O will be non-decreasing and the distance between O and H will be non-increasing. This means that the distance from O to the boundary of P will be non-increasing. This contradicts Lemma 4. □

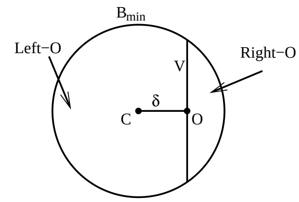

Figure 1: Partition B_{min} into *Left*-O and *Right*-O.

From Lemma 3, we know that to find the MaxIS of $\mathcal{H}$, it is sufficient to make the origin O and the center C of the MinES coincide. To reach this goal, our main idea is to move the origin O within the polytope P and force the center C to move accordingly so that their distance become smaller and smaller.

Before we further discuss how to move O, we first introduce some notations. Without loss of generality, we assume that the segment $\overline{OO'}$ is horizontal, and O is to the left of O' (see Figure 1). Let V be the hyperplane crossing O and orthogonal to $\overline{OO'}$. V partitions B_{min} into two parts. We denote the left part of B_{min} as $Left - O$ sphere and the right part as *Right*-O sphere. Clearly,

$$\textit{Left}\text{-O sphere} = \{x \mid \|f(x)\| < \|x\|, x \in B_{min}\}, \text{ and}$$

$$\textit{Right}\text{-O sphere} = \{x \mid \|f(x)\| > \|x\|, x \in B_{min}\}.$$

From Lemma 2 and the dual transform, it is easy to see that in order to make C and O coincide (i.e., $\delta = 0$), O has to move in certain direction. Since when C overlaps O, MinES has the smallest radius, this means that forcing C to overlap O is equivalent to minimize the radius of MinES or the longest distance from O to a point in $\mathcal{H}^*$. Since the farthest point (to O) can only appear in the *Left*-O sphere, to ensure that the movement of O always reduces the longest distance to O, O has to move away from C, ideally in the direction of $\overline{CO}$. The following lemmas show some nice properties of such a movement.

LEMMA 6. *If O, x, y are collinear, and both x and y are in Left-O (or Right-O) sphere, then $\|x\| \leq \|y\|$ implies $\|f(x)\| < \|f(y)\|$.*

Lemma 6 shows that to estimate how the MinES changes after moving O to O', it is sufficient to consider only the points on the boundary of the MinES.

LEMMA 7. *Let x be any point in V, then $\|f(x)\| = \|x\|$.*

LEMMA 8. *Let r_{opt} be the radius of the MinES when C overlaps O, and R_{opt} be the radius of the MaxIS of $\mathcal{H}$. Let r be the radius of the MinES B_{min} (whose center C may not overlap O), and R be the shortest distance from O to the hyperplanes in P. If $r \leq (1+\epsilon)r_{opt}$ and $\delta \leq \epsilon r_{opt}$, then $R_{opt} \geq R \geq (1-2\epsilon)R_{opt}$.*

PROOF. When $r \leq (1+\epsilon)r_{opt}$ and $\delta \leq \epsilon r_{opt}$, the furthest point in $\mathcal{H}^*$ to O is no more than $(1+2\epsilon)r_{opt}$. Thus,

$$R \geq \frac{1}{(1+2\epsilon)r_{opt}} \geq (1-2\epsilon)R_{opt}.$$

□

Lemma 8 suggests that to compute a $(1-\epsilon)$-approximation of the MaxIS, it is sufficient to reduce the radius r of the MinES so that $r \leq (1+\epsilon/2)r_{opt}$ and $\delta \leq \epsilon r_{opt}/2$. Also note that, for any $\epsilon_0 > 0$ we can always choose $\epsilon = \epsilon_0/2$ and compute a $(1+\epsilon)$-approximation of the r_{opt} and make $\delta \leq \epsilon r_{opt}$. This will ensure that the obtained solution is a $(1-\epsilon_0)$-approximation of R_{opt}. Hence, hereafter we will focus on computing a $(1+\epsilon)$-approximation of r_{opt} and on reducing δ.

3. ALGORITHM

As shown in last section, the MaxIS problem can be reduced to solving the MinES problem. Thus our main focus in this section is on how to move O so that the radius r of the MinES can be quickly reduced to the level of $(1+\epsilon)r_{opt}$ and δ be reduced to ϵr_{opt}.

From Lemma 5, we know that the farthest point of O lies in the half-sphere which is to the left of the hyperplane crossing C and orthogonal to $\overline{CO}$. Since the farthest point could be in any place of the boundary of this half-sphere, to ensure that the longest distance to O will always be reduced, we move O in the direction of $\overline{CO}$.

Once determined the motion direction, we immediately face two more questions: (1) How much should O move in the direction of $\overline{CO}$ (i.e., what is the value of $\|s\|$)? (2) How much is the radius r of the MinES reduced (i.e., what is the value of r/r')? Clearly, exact answers to the two questions depend on the distribution of $\mathcal{H}^*$ inside the MinES. To simplify our task, instead of giving an exact answer to each question, we estimate the values of $\|s\|$ and r/r'. More specifically, our idea is to ignore the exact distribution of the points of $\mathcal{H}^*$ inside MinES and consider only the boundary ∂B_{min} of the MinES B_{min}. By Lemma 6, we know that after moving O to O', all points in $\mathcal{H}^{*'}$ will still be inside the region bounded by $f(\partial B_{min})$. (Note that $f(\partial B_{min})$ is no longer a sphere.) Thus, by estimating the radius of the minimum sphere B^f_{min} enclosing $f(\partial B_{min})$, we can obtain the upper bound of the radius r'. The following lemma shows some property of B^f_{min}.

LEMMA 9. *Both $f(\partial B_{min})$ and B^f_{min} are symmetric about $\overline{CO}$ and the center C^f of B^f_{min} is on the segment of $\overline{CO}$.*

Clearly, to get the maximum reduction on r, O should be moved as much as possible so that the farthest point to O' in $f(\partial B_{min})$ is minimized. However, by Lemma 7, we know that points in $V \cap \partial B_{min}$ are fixed points (i.e., their distances to the origin do not change). Thus our main idea is to see if we can move O so that points in $V \cap \partial B_{min}$, which are $\sqrt{r^2-\delta^2}$ distance away from O, become the farthest points to O'. The following lemma shows that such a movement exist and the function $\|f(x)\|$ has some monotonicity on the boundary of B_{min}.

LEMMA 10. *If $\|s\| = \frac{\delta}{r^2-\delta^2}$, then for any point x on the boundary ∂B_{min} of the MinES, $\|f(x)\| < \sqrt{r^2-\delta^2}$. Further, (1) for any pair of points x and y on the boundary of Left-O sphere, $\|x\| > \|y\|$ implies $\|f(x)\| < \|f(y)\|$; and (2) for any pair of points x and y on the boundary of Right-O sphere, $\|x\| > \|y\|$ implies $\|f(x)\| > \|f(y)\|$.*

PROOF. We prove only for the boundary of *Right*-O sphere. The proof for the boundary of *Left*-O sphere is easier and omitted.

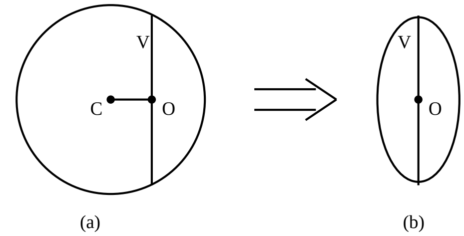

Figure 2: Illustration of a primitive move.

Let x and y be any pair of points on the boundary of *Right*-O sphere. Since $f(x) = \frac{x}{1-x\cdot s}$,

$$\|f(x)\| = \frac{\|x\|}{1-\|x\|\frac{\delta}{r^2-\delta^2}\cos\theta}, \quad (1)$$

where θ is the angle between Ox and OO'. By the definition of θ, we have

$$\cos\theta = \frac{r^2-\delta^2-\|x\|^2}{2\delta\|x\|}.$$

Plugging $\cos\theta$ into (1), we get

$$\|f(x)\| = \frac{2\|x\|(r^2-\delta^2)}{r^2-\delta^2+\|x\|^2}.$$

Similarly, we have

$$\|f(y)\| = \frac{2\|y\|(r^2-\delta^2)}{r^2-\delta^2+\|y\|^2}.$$

Since $\|x\| > \|y\|$, we have

$$\|x\|(r^2-\delta^2) + \|x\|\|y^2\| - \|y\|(r^2-\delta^2) - \|y\|\|x^2\|$$

$$= (\|x\|-\|y\|)(r^2-\delta^2) + \|x\|\|y\|(\|y\|-\|x\|)$$

$$= (\|x\|-\|y\|)(r^2-\delta^2-\|x\|\|y\|).$$

Since $\|x\|, \|y\| \in (r-\delta, \sqrt{r^2-\delta^2})$, the above equation is positive. This implies $\|f(x)\| > \|f(y)\|$.

Also since the point p in $V \cap \partial B_{min}$ is farther away from O than any other point in the boundary of *Right*-O sphere and $\|f(p)\| = \sqrt{r^2-\delta^2}$, thus $\|f(x)\| \leq \sqrt{r^2-\delta^2}$. □

Lemma 10 suggests that to have the maximum reduction on r, O should move in the direction of $\vec{CO}$ by a distance of $\|s\| = \frac{\delta}{r^2-\delta^2}$, and the new radius r' of the MinES of $\mathcal{H}^{*'}$ is no more than $\sqrt{r^2-\delta^2}$. We call such a move as a *primitive move* (see Figure 2).

Once determined how to move O in one step, a naive idea is just to repeatedly perform the primitive moves on O and reduce r so that r will eventually be smaller than $(1+\epsilon)r_{opt}$. In this way, we only need to bound the total number of steps. Unfortunately, since the interior point O could be in any place of the polytope P and P could be of arbitrary (convex) shape, the total number of primitive moves is not always a constant. To overcome this difficulty, we first partition the problem into two cases: (A) the starting point of O is in a "good" position; (B) the starting point of O is in arbitrary position. Below we first consider case (A).

We say O is in a "good" position if the initial radius r_0 of the MinES B_{min} is at most constant times of r_{opt}, i.e., $r_0 \leq k \cdot r_{opt}$ for some constant k. To bound the total number of steps needed for this case, we first assume we can compute in each step the exact MinES of $\mathcal{H}^*$, and then show how to use core-sets to speed up the computation.

LEMMA 11. *If $r \leq (1+\epsilon)r_{opt}$, then either (1) $\delta \leq \epsilon r_{opt}$ or (2) if $\delta > \epsilon r_{opt}$, then after at most $1/\epsilon$ primitive moves, δ can be reduced to ϵr_{opt}.*

PROOF. Since the first case is trivial, we focus on the second case.

Assume that $\delta > \epsilon r_{opt}$. By the above discussion about the primitive move, we know that if we perform a primitive move on O, the new radius r' of the MinES is

$$r' = \sqrt{r^2 - \delta^2} < \sqrt{r^2 - (\epsilon r_{opt})^2}.$$

Using Taylor's expansion, we have

$$r' < r - \frac{(\epsilon r_{opt})^2}{2r} \leq r - \frac{(\epsilon r_{opt})^2}{(1+\epsilon)r_{opt}}$$

$$< r - \epsilon^2 r_{opt}.$$

By the assumption, we know that $r - r_{opt} \leq \epsilon r_{opt}$ and each primitive move reduces r by at least $\epsilon^2 r_{opt}$. If δ is always larger than ϵr_{opt}, in $1/\epsilon$ steps r will be reduced to r_{opt}. By Lemma 3, we know that at that time $\delta = 0$. Thus the lemma follows. □

Combining Lemmas 8 and 11, we know that to obtain a $(1-\epsilon)$-approximation of R_{opt}, it is sufficient to focusing on reducing r.

Let α be the aspect ratio of P (i.e., the ratio of the radii of the minimum enclosing sphere of P and the MaxIS of P). In this paper we consider the case that α is a constant.

LEMMA 12. *For a given $\epsilon_0 > 0$, let $\epsilon = \frac{\epsilon_0}{\frac{4}{3}\alpha}$. Then, if $\delta < r_{opt}\epsilon$, $r < (1+3\epsilon_0)r_{opt}$.*

To prove this lemma, we need the following lemma.

LEMMA 13. *Let O and C be respectively the origin and the center of the MinES with $\delta < r_{opt}\epsilon$ and $r > (1+3\epsilon)r_{opt}$. Let O_{opt} be the closest optimal (i.e., Chebyshev) center and V_{opt} be the hyperplane with $O \in V_{opt}$ and $V_{opt} \perp \overline{OO_{opt}}$. Then, no point x is on the boundary of the MinES with $x \cdot OO_{opt} > 0$.*

PROOF. Suppose there is such an x on the boundary of the MinES with $x \cdot OO_{opt} > 0$. Below we show that this would lead to a contradiction. Our main idea is to demonstrate that if there is such an x, then after moving O to O_{opt}, the distance from O_{opt} to the corresponding hyperplane of x would be smaller than R_{opt}, thus a contradiction.

First we know that in this case, $r_{opt} < r \leq kr_{opt}$. Let $r = k' r_{opt}$ for some $k' \in (1+\lambda\epsilon, k]$ and $\lambda \geq 3$. Then,

$$x > r - r_{opt}\epsilon = (k' - \epsilon)r_{opt} > r_{opt}, \text{when } k' > 1+\lambda\epsilon.$$

Next we show that when $\lambda \geq 3$, a contradiction can be derived. Let H be the corresponding hyperplane of x, then the distance from O to H is

$$\|H\| = \frac{1}{\|x\|} < \frac{1}{r - r_{opt}\epsilon} = \frac{1}{k' - \epsilon} \cdot R_{opt}$$

$$= \frac{1}{k'} \frac{1}{1 - \frac{\epsilon}{k'}} R_{opt} < \frac{1}{k'}(1 + \frac{2\epsilon}{k'})R_{opt}.$$

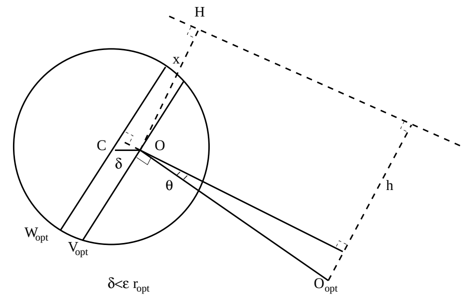

Figure 3: Illustration of Lemma 12.

To ensure a contradiction, λ has to satify the following inequality.

$$\frac{1}{k'}(1 + \frac{2\epsilon}{k'}) < 1 - \epsilon. \quad (2)$$

This means

$$1 + \frac{2\epsilon}{k'} < k'(1-\epsilon),$$

$$1 + \frac{2\epsilon}{k'} < (1+\lambda\epsilon)(1-\epsilon) = 1 - \epsilon + \lambda\epsilon - \lambda\epsilon^2,$$

$$\frac{2}{k'} < \lambda - 1 - \lambda\epsilon,$$

$$\frac{2/k' + 1}{1-\epsilon} < \lambda.$$

Notice that from $k' > 1 + \lambda\epsilon$, we have

$$2/k' \leq 2/(1+\lambda\epsilon). \quad (3)$$

Thus,

$$\frac{\frac{2}{1+\lambda\epsilon} + 1}{1-\epsilon} < \lambda. \quad (4)$$

This implies

$$\frac{2}{1+\lambda\epsilon} + 1 < \lambda(1-\epsilon),$$

$$2 + 1 + \lambda\epsilon < \lambda(1-\epsilon)(1+\lambda\epsilon),$$

$$3 + \lambda\epsilon < \lambda(1 + \lambda\epsilon - \epsilon - \lambda\epsilon^2). \quad (5)$$

It is easy to see that when $\lambda \geq 3$, (5) holds.

This means that when $k' > 1 + 3\epsilon$,

$$\frac{1}{k'}(1 + \frac{2\epsilon}{k'})R_{opt} < (1-\epsilon)R_{opt}. \quad (6)$$

Now we have $\|H\| < (1-\epsilon)R_{opt}$. Thus if moving O to O_{opt}, the distance from O to H will be shorter than the current $\|H\|$, which contradicts to the fact that R_{opt} is the radius of the MaxIS. □

Now we prove Lemma 12.

PROOF. We prove this lemma by contradiction. First, from Lemma 13, we know that there is no point x on the boundary of the MinES with $x \cdot OO_{opt} > 0$. Suppose the lemma is not true. Then, we assume $r > (1+3\epsilon_0)r_{opt}$.

By Lemma 5, we know that there is at least one boundary point, say x, between V_{opt} and W_{opt}, where V_{opt} and O_{opt} are defined as in Lemma 13 and W_{opt} is the hyperplane containing C and orthogonal to $\overline{OO_{opt}}$.

Next we lower bound the distance $\|OO_{opt}\|$ between O and O_{opt}. Let H be the corresponding hyperplane of x, and h be the distance from O_{opt} to H after moving O to O_{opt}. Then we have

$$h_{>}R_{opt} \tag{7}$$

From Figure 3, it is easy to see that

$$\|OO_{opt}\| \geq \frac{h - \frac{1}{\|x\|}}{\sin\theta}. \tag{8}$$

Since $\delta < \epsilon r_{opt}$, we can upper bound $\sin\theta$ as follows.

$$\max\sin\theta \leq \frac{r_{opt}\epsilon}{\min\|x\|} \leq \frac{r_{opt}\epsilon}{r - \epsilon r_{opt}}$$

$$< \frac{r_{opt}\epsilon}{(1+3\epsilon_0)r_{opt} - \epsilon r_{opt}} \leq \frac{\epsilon}{1+2\epsilon}$$

Thus $\|OO_{opt}\|$ can be lower bounded by

$$\|OO_{opt}\| \geq \frac{h - \frac{1}{r - r_{opt}\epsilon_0}}{\sin\theta} \tag{9}$$

By assumption, we have

$$r - r_{opt}\epsilon_0 > r_{opt} + 2r_{opt}\epsilon_0. \tag{10}$$

Thus,

$$\frac{1}{r_{opt} + 2\epsilon_0 r_{opt}} < R_{opt}(1 - \frac{3}{2}\epsilon_0), \text{when } \epsilon_0 < \frac{1}{6}.$$

Plugging the above inequality into (9), we have

$$\|OO\| > \frac{R_{opt}(1 - 1 + \frac{3}{2}\epsilon_0)(1+2\epsilon)}{\epsilon}, \text{and}$$

$$\|OO_{opt}\| > R_{opt}\frac{3}{2}\frac{4}{3}\alpha(1+2\epsilon) > 2R_{opt}\alpha.$$

But we know that $\|OO_{opt}\| < 2R_{opt}\alpha$ (from the definition of aspect ratio). A contradiction. This means that when $\delta < \epsilon r_{opt}, r < (1+3\epsilon_0)r_{opt}$. □

The above lemma suggests that given an ϵ_0, by using a smaller ϵ, we can ensure that when $r \geq (1+3\epsilon_0)r_{opt}$, $\delta > r_{opt}\cdot\epsilon$. Thus in each step, the converging ratio of r is

$$\frac{(r - \sqrt{r^2 - \delta^2})}{r} = 1 - \sqrt{1 - (\frac{\delta}{r})^2}$$

$$\geq 1 - \sqrt{1 - (\frac{r_{opt}\cdot\epsilon}{r})^2}$$

$$\geq 1 - \sqrt{1 - (\frac{r_{opt}\cdot\epsilon}{k\cdot r_{opt}})^2}$$

$$= 1 - \frac{1}{k}\cdot\sqrt{k^2 - \epsilon^2}$$

$$= \frac{k - \sqrt{k^2 - \epsilon^2}}{k}.$$

The total number of steps is no more than

$$\frac{r}{r_{opt}\frac{k - \sqrt{k^2-\epsilon^2}}{k}} = \frac{k}{1 - \sqrt{1 - (\frac{\epsilon}{k})^2}}$$

$$< \frac{k}{1 - (1 - \frac{1}{2}(\frac{\epsilon}{k})^2)} = 2\frac{k^3}{\epsilon^2}.$$

Since k is a constant, the total number of steps is $O(1/\epsilon^2)$.

Lemma 14. *If MinES is exactly computed in each step, then the total number of primitive moves for case (a) is* $O(1/\epsilon^2)$ *(i.e., when k is a constant).*

Now we consider using core-sets to compute the MinES. Let r be the radius of the exact MinES in each step, and r_a be the radius of an approximation of the MinES (using algorithm in [4, 16]) in the same step.

Lemma 15. *If a $(1+\epsilon/k)$-approximation of the MinES is computed in each step, then the total number of primitive moves needed for case (A) is bounded by $O(1/\epsilon^2)$, where k is the constant in the definition of case (A).*

Proof. Observe the above proof for bounding the total number of primitive moves for case (A). The proof does not require r be the radius of the exact MinES of $\mathcal{H}^*$. It is actually sufficient to bound the total number of primitive moves by $O(1/\epsilon^2)$, as long as the reduction on the radius r at each step is no less than $r - \sqrt{r^2 - \delta^2}$.

Consider the case of using a $(1+\epsilon/k)$-approximation algorithm to compute the MinES at each step. By the above discussion, we know that the total number of steps will still be $O(1/\epsilon^2)$ if at each step the radius r'_a returned by the approximation algorithm after a primitive move is no larger than $\sqrt{r_a^2 - \delta^2}$, where r_a is the radius of the approximate MinES computed at the previous step.

From the discussion about the primitive move, we know that for any dual point H^*, $\|H^*\| \leq \sqrt{r_a^2 - \delta^2}$. This means that a ball B centered at O and with radius $\sqrt{r_a^2 - \delta^2}$ contains all points in $\mathcal{H}^{*'}$, and the radius r' of the exact MinES is no more than $\sqrt{r_a^2 - \delta^2}$. Thus, the only chance for the approximation algorithm to generate a ball B_a with radius $r'_a > \sqrt{r_a^2 - \delta^2}$ is when $r' > \frac{\sqrt{r_a^2 - \delta^2}}{1+\epsilon/k}$. But in this case, the distance between the center C_a of B_a and O will be less than $\frac{\epsilon\sqrt{r_a^2 - \delta^2}}{k}$, and δ will be less than ϵr_{opt}. This means that the algorithm can stop. □

Lemmas 14 and 15 indicate that if O is in a good position, then the total number of steps needed to reduce r_0 to $(1+\epsilon)r_{opt}$ is no more than $O(1/\epsilon^2)$, and each step takes $O(nd/\epsilon)$ time.

Next we consider case (B) (i.e., O is initially in an arbitrary position). From Lemma 1, we know that given an arbitrary polytope P and an arbitrary interior point O, since O could be very close to a boundary of P, the initial value of r_0 could be arbitrarily larger than r_{opt}. Thus if we just perform the above procedure on O, the number of steps will depend on the initial position of the origin. Next we show how to remove this dependency.

Let h_i be the distance from O to the hyperplane $H_i \in \mathcal{H}, i = 1, \cdots, n$.

Lemma 16. *In dual space, the radius r of the MinES is a constant times of r_{opt} if and only if $h_{min} = \min_{i=1}^{n}\{h_i\}$ is a constant times of R_{opt}.*

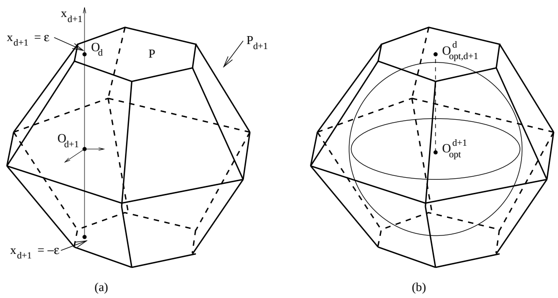

Figure 4: Illustration of P_{d+1}.

We first use affine scaling to make $R_{opt} \leq \epsilon$. Let $h_{max} = max_{i=1}^{n}\{h_i\}$ and $u = \frac{h_{max}}{\epsilon}$. Then the coordinate system is scaled in the following way

$$x_i = \frac{x_i}{u}, i = 1, \cdots, d.$$

It is easy to prove that after this scaling, $R_{opt} \leq \epsilon$, and the cost of the scaling is $O(nd)$. From now on, we assume that $R_{opt} \leq \epsilon$.

To move O away from the boundary of P, we first introduce a new dimension x_{d+1} orthogonal to all other dimensions $x_i, 1 \leq i \leq d$, and then build a new polytope P_{d+1} in $(d+1)$-dimensional space E^{d+1}. The polytope P_{d+1} is defined by $2n+2$ hyperplanes (see Figure 4(a)).

$$P_{d+1} = \{((H_1, \epsilon), \epsilon + h_1), ((H_1, -\epsilon), -\epsilon - h_1)), \cdots,$$

$$((H_n, \epsilon), \epsilon + h_n)), ((H_n, -\epsilon), -\epsilon - h_n)), \Phi_1, \Phi_2\},$$

where $\Phi_1 = \{x \in E^{d+1} | x_{d+1} = \epsilon\}$, $\Phi_2 = \{x \in E^{d+1} | x_{d+1} = -\epsilon\}$, and $((H_i, \epsilon), \epsilon + h_i))$ represents a $(d+1)$-dimensional hyperplane which contains a point $(0, ..., 0, \epsilon + h_i)$ on the x_{d+1} axis and intersects with Φ_1 at the d-dimensional hyperplane H_i.

Let O_d be any interior point of $P \cap \Phi_1$, and O_{d+1} by its projection (along x_{d+1} axis) on the hyperplane $x_{d+1} = 0$. Without loss of generality, we assume that O_{d+1} is the origin of E^{d+1}. Let O_{opt}^{d+1} and O_{opt} be the optimal points of P_{d+1} and P respectively, R_{opt}^{d+1} and R_{opt} be the respective radii of the MaxIS's of P_{d+1} and P. By the symmetry of P_{d+1}, it is easy to see that O_{opt}^{d+1} is on the hyperplane of $x_{d+1} = 0$. The following lemmas shows some interesting properties of P_{d+1}.

Lemma 17. *Let H be any hyperplane in P and H_{d+1} be one of its corresponding hyperplane $((H_i, \epsilon), \epsilon + h_i)) \in P_{d+1}$. Then $dist(O_{d+1}, H_{d+1}) = \frac{\sqrt{2}}{2}(\epsilon + dist(O_d, H))$, where $dist()$ is the Euclidean distance from one point to a hyperplane.*

Proof. Assume that the hyperplane H is of the form

$$a_1x_1 + a_2x_2 + ... + a_dx_d = 1. \quad (11)$$

In E^{d+1}, the corresponding hyperplane H_{d+1} has the form

$$a_1'x_1 + ... + a_d'x_d + a_{d+1}'x_{d+1} = 1. \quad (12)$$

Let $T = \sqrt{a_1^2 + a_2^2 + ... + a_d^2}$. Then the distance $dist(O_d, H)$ between O_d and H is $1/T$. From the construction of P_{d+1}, we have

$$a_{d+1}' = \frac{1}{\frac{1}{T} + \epsilon} = \frac{T}{1 + T \cdot \epsilon}. \quad (13)$$

When restricting $x_{d+1} = \epsilon$, H and H_{d+1} are the same, i.e., (12) = (11). Thus, we have

$$a_1'x_1 + ... + a_d'x_d = 1 - \frac{T\epsilon}{1 + T\epsilon}, \text{and}$$

$$a_i = a_i'(1 + T\epsilon), \text{for } i = 1, \cdots, d. \quad (14)$$

The distance from O_{d+1} to H_{d+1} is

$$dist(O_{d+1}, H_{d+1}) = \frac{1}{\sqrt{{a_1'}^2 + ... + {a_d'}^2 + {a_{d+1}'}^2}}$$

$$= \frac{1}{\sqrt{T^2(\frac{1}{1+T\epsilon})^2 + (\frac{T}{1+T\epsilon})^2}}$$

$$= \frac{1}{\sqrt{2(\frac{T}{1+T\epsilon})^2}} = \frac{\sqrt{2}}{2}(\epsilon + (\frac{1}{T})) \quad (15)$$

$$= \frac{\sqrt{2}}{2}(\epsilon + dist(O_d, H)).$$

□

Lemma 18. $\frac{\sqrt{2}\epsilon}{2} \leq R_{opt}^{d+1} \leq \epsilon$.

Proof. The first inequality follows from Lemma 17, since the closest distance from the $(d+1)$-dimensional origin O_{d+1} to any hyperplane is at least $\frac{\sqrt{2}}{2} \times \epsilon$. The second inequality follows from the fact that the distance between Φ_1 and Φ_2 is exact 2ϵ. □

Lemma 19. *For any O_d in $P \cap \Phi_1$, the projection of O_d to the hyperplane $x_{d+1} = 0$ is a good starting point for P_{d+1}.*

Proof. From Lemma 18, we know that $R_{opt}^{d+1} \leq \epsilon$. From Lemma 17, we know that for any interior point O_d, the projection O_{d+1} to the hyperplane $x_{d+1} = 0$ is at least $\frac{\sqrt{2}\epsilon}{2}$ away from any hyperplane in P_{d+1}. Thus the radius of the maximum inscribed sphere centered at O_{d+1} is a constant times of R_{opt}^{d+1}. This implies that O_{d+1} is a good starting point for P_{d+1}. □

Lemma 20. *The projection point $O_{opt,d+1}^{d}$ of O_{opt}^{d+1} on Φ_1 is an interior point of P, i.e. $O_{opt,d+1}^{d} \in P$ (see Figure 4(b)).*

Proof. We consider two cases, (a) $\epsilon < \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$ and (b) $\epsilon \geq \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$. Case (a) follows trivially from Lemma 17. Thus we only need to focus on case (b). We prove this case by contradiction.

Suppose $O_{opt,d+1}^{d} \notin P$, then by Lemma 17, it is easy to see that

$$R_{opt}^{d+1} \leq \frac{\sqrt{2}}{2}\epsilon < \frac{\sqrt{2}}{2}(\epsilon + R_{opt}^{d}).$$

Let $O_{opt,d}^{d+1}$ be the projection of O_{opt} (i.e., the optimal point of P) on the hyperplane $x_{d+1} = 0$, then using $O_{opt,d}^{d+1}$ as the center and $\frac{\sqrt{2}}{2}(\epsilon + R_{opt})$ as the radius, we can construct a ball B. By Lemma 17, we know $B \subset P_{d+1}$. Thus, $R_{opt}^{d+1} \geq \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$, a contradiction. □

LEMMA 21. *Let $O^d_{opt,d+1}$ be the projection of O^{d+1}_{opt} on Φ_1. Then, the minimum distance h_{min} from $O^d_{opt,d+1}$ to any hyperplane in P is at least $\frac{1}{3}R_{opt}$ (i.e., $O^d_{opt,d+1}$ is a good starting point for P).*

PROOF. Let B_{opt} be the MaxIS of P centered at O_{opt} and $S_P(O_{opt}) = \{H|\ H \in P \text{ is tangent to } B_{opt}\}$. $S_P(O_{opt})$ is called the solution set of O_{opt}.

Next we consider two cases, (a) $\epsilon < \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$ and (b) $\epsilon \geq \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$. We first consider case (b).

For case (b), it is easy to see that if we choose the solution set $S_P(O_{opt})$ of $O_{opt} \in P$, then we can construct a ball B centered at the projection point of O_{opt} on the hyperplane $x_{d+1} = 0$ and with radius $R = \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$. Clearly, $B \subset P_{d+1}$. B might be tangent to any hyperplane in P_{d+1} except Φ_1 and Φ_2. Thus $R^{d+1}_{opt} \geq R$.

Let $S_{P_{d+1}}(O^{d+1}_{opt})$ be the solution set of O^{d+1}_{opt}. From Lemma 20, we know that $O^d_{opt,d+1} \in P$. From Lemma 17, we know that the closest hyperplanes to $O^d_{opt,d+1}$ in P are those corresponding to the hyperplanes in $S_{P_{d+1}}(O^{d+1}_{opt})$. Let h_{min} be the closest distance between $O^d_{opt,d+1}$ and the hyperplanes in P. By Lemma 17, we have

$$R^{d+1}_{opt} = \frac{\sqrt{2}}{2}(\epsilon + h_{min}).$$

Thus,

$$R^{d+1}_{opt} = \frac{\sqrt{2}}{2}(\epsilon + h_{min}) \geq \frac{\sqrt{2}}{2}(\epsilon + R_{opt}),$$

$$h_{min} \geq R_{opt}.$$

Since R_{opt} is the radius of the MaxIS of P, R_{opt} can only be equal to h_{min}. This means that $O^d_{opt,d+1} = O_{opt}$, i.e., the projection $O^d_{opt,d+1}$ of O^{d+1}_{opt} on Φ_1 is an optimal point of P, and $O^d_{opt,d+1}$ is trivially a good starting point of P.

For case (a) (i.e., $\epsilon < \frac{\sqrt{2}}{2}(\epsilon + R_{opt})$), from Lemma 17 we know that the hyperplanes in P^{d+1} corresponding to the solution set of O_{opt} are the closest hyperplanes to $O^{d+1}_{opt,d}$ (except Φ_1 and Φ_2), where $O^{d+1}_{opt,d}$ is the projection of O_{opt} on the hyperplane $x_{d+1} = 0$. Since we already know that these hyperplanes have a distance $\frac{\sqrt{2}}{2}(\epsilon + R_{opt})$ to $O^{d+1}_{opt,d}$, it is easy to construct a ball B with radius $R = \epsilon$, centered at $O^{d+1}_{opt,d}$, and inside P_{d+1}. So $R^{d+1}_{opt} \geq \epsilon$. By Lemma 18, we also know that $R^{d+1}_{opt} \leq \epsilon$, thus $R^{d+1}_{opt} = \epsilon$.

Note that in this case, the MaxIS of P_{d+1} may not be unique. We can compute O^{d+1}_{opt} first and then consider the location of $O^d_{opt,d+1}$. Below we prove that $O^d_{opt,d+1}$ is very close to some O_{opt}.

Let t be the distance from $O^d_{opt,d+1}$ to the closest hyperplane H in P, and H_{d+1} be the corresponding hyperplane of H in P_{d+1}. Then the MaxIS of P_{d+1} could be either tangent to H_{d+1} or not. In either case, we have

$$\frac{\sqrt{2}}{2}(\epsilon + t) \geq \epsilon.$$

Thus,

$$t \geq (\sqrt{2} - 1)\epsilon. \tag{16}$$

This means that $O^d_{opt,d+1}$ is a good starting point as $R_{opt} \leq \epsilon$. Here,

$$k = \frac{1}{\sqrt{2} - 1} < 3.$$

□

The above lemmas suggest that when the starting point O_d is not good, we can first project O_d to the hyperplane $x_{d+1} = 0$ to obtain O_{d+1}, perform the algorithm for case (A) in P_{d+1} to obtain the optimal (or near optimal) point of P_{d+1}, and then project the optimal point of P_{d+1} back to P. The point we obtained is either an optimal point of P or a good starting point of P. Below we show that if an approximation algorithm is used to compute the MaxIS of P_{d+1}, it is possible to obtain a good starting point in P by performing a constant number of iterations in $d+1$-dimensional space, each with a slightly different P_{d+1}.

The main difficulty of using an approximation algorithm is that when $R_{opt} \ll \epsilon$, a $(1-\epsilon)$-approximation of the MaxIS of P_{d+1} may not be a good starting point of P. To overcome this difficulty, our main idea is to use the following observation: When $2R_{opt}$ is roughly equal to the height of P_{d+1} (along the x_{d+1}-axis), the projection of the center of a $(1-\epsilon)$-approximation of P_{d+1} is a good starting point of P. Thus our idea is to guess R_{opt} and construct a P_{d+1} polytope with x_{d+1}-height equal to twice of the guessed value of R_{opt}. Since ϵ (equal to the distance from O to the farthest hyperplane in P after the affine scaling) can only be 2α times of R_{opt}, by performing a binary search (with at most $\log \alpha$ steps which is a constant when α is bounded) in the interval $[\frac{\epsilon}{2\alpha}, \epsilon]$, we can find a good starting point for P. The following lemma summarizes the above discussion. (Detailed proof is left for the full paper.)

LEMMA 22. *Within $\log \alpha$ iterations, a good starting point of P can be founded by using the approximation algorithm for case (A) in P_{d+1}.*

With the above lemmas, we can put all the pieces together and have the following main steps for our algorithm for computing a $(1-\epsilon_0)$-approximation of the MaxIS of P.

1. Set $\epsilon = \epsilon_0/(8\alpha)$.
2. Translate the origin of the coordinate system to the interior point O.
3. Scale the coordinate system so that $R_{opt} \leq \epsilon$.
4. Perform a dual transform for each halfspace in $\mathcal{H}$ to obtain $\mathcal{H}^*$.
5. Build the $d+1$ dimensional polytope P_{d+1}.
6. Perform the $(1-\epsilon)$-approximation for the (good) case (A) $\log(\alpha)$ iterations on P_{d+1}'s, and project the obtained center back to P.
7. Using the projection as the starting point, run the $(1-\epsilon)$-approximation for case (A) on P and return the obtained solution.

THEOREM 1. *Given any polytope defined by n d-dimensional halfspaces (or linear constraints) and with bounded aspect ratio α, an interior point, and a small constant $0 < \epsilon < 0.1$, there exists an approximation algorithm which computes a $(1-\epsilon)$-approximation of the maximum inscribed sphere of the polytope in $O(nd/\epsilon^3)$ time.*

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