

- 1-
- a)  $\forall x (\text{Member}(x) \equiv x = \text{Joe} \vee x = \text{Sally} \vee x = \text{Bill} \vee x = \text{Ellen})$  ①  
 $\text{MarriedTo}(\text{Joe}, \text{Sally})$  ② 1  
 $\text{Brother}(\text{Ellen}, \text{Bill})$  ③ 1  
 $\forall x \forall y (\text{Member}(x) \wedge \text{MarriedTo}(x, y) \supset \text{Member}(y))$  ④

b) We show that this set S does not entail  $\neg \exists x \text{MarriedTo}(\text{Ellen}, x)$

Take  $\mathcal{A} = \langle D, I \rangle$

$D = \{ \text{Joe}, \text{Sally}, \text{Bill}, \text{Ellen} \}$

$I(\text{Joe}) = \text{Joe}$  & similarly for other constants

$I(\text{Member}) = D$

$I(\text{MarriedTo}) = \{ \langle \text{Joe}, \text{Sally} \rangle, \langle \text{Sally}, \text{Joe} \rangle, \langle \text{Bill}, \text{Ellen} \rangle, \langle \text{Ellen}, \text{Bill} \rangle \}$

$I(\text{Brother}) = \{ \langle \text{Ellen}, \text{Bill} \rangle \}$

Then  $\mathcal{A} \models S$  but  $\mathcal{A} \not\models \neg \exists x \text{MarriedTo}(\text{Ellen}, x)$

c) Add

$\forall x \forall y (\text{Brother}(x, y) \supset \neg \text{MarriedTo}(x, y))$  ⑤

$\forall x \forall y (\text{MarriedTo}(x, y) \supset \text{MarriedTo}(y, x))$  ⑥

$\forall x (\neg \text{MarriedTo}(x, x))$  ⑦

$\forall x \forall y \forall z (\text{MarriedTo}(x, y) \wedge \text{MarriedTo}(x, z) \supset y = z)$  ⑧

$x \neq y$  for distinct  $x, y \in \{ \text{Joe}, \text{Sally}, \text{Bill}, \text{Ellen} \}$

⑨ UNA

Call this  $S'$

Take a b  $\mathcal{A} = \langle \mathcal{D}, \mathcal{I} \rangle$  assume  $\mathcal{A} \models S'$ ,

By ① and UNA  $\mathcal{I}(\text{Member}) = \{ \mathcal{I}(\text{Jae}), \mathcal{I}(\text{Sally}), \mathcal{I}(\text{Bill}), \mathcal{I}(\text{Ellen}) \}$

By ② and ⑧ and UNA  $\mathcal{A} \models \neg \text{MarriedTo}(\text{Ellen}, \text{Sally})$

By ② and ⑥ and ⑧ and UNA  $\mathcal{A} \models \neg \text{MarriedTo}(\text{Ellen}, \text{Jae})$

By ⑤ and ③  $\mathcal{A} \models \neg \text{MarriedTo}(\text{Ellen}, \text{Bill})$

By ⑦  $\mathcal{A} \models \neg \text{MarriedTo}(\text{Ellen}, \text{Ellen})$

Thus  $\mathcal{A} \models \forall x (\text{Member}(x) \supset \neg \text{MarriedTo}(\text{Ellen}, x))$  (\*)

Suppose there is  $a \in \mathcal{D}$  st  $\mathcal{A} \models \text{MarriedTo}(\text{Ellen}, x)$   $\mathcal{U}(x) = a$  (Ass)

we have  $\mathcal{A} \models \text{Member}(\text{Ellen})$  by ①

so by ④  $\mathcal{A}, \mathcal{U} \models \text{Member}(x)$

This together with \* contradicts (Ass)

So  $S' \models \neg \exists x \text{ MarriedTo}(\text{Ellen}, x)$

2-   
 Show

$$\models \exists x \forall y \forall z ((P(y) \supset Q(z)) \supset (P(x) \supset Q(x)))$$

Negate query

$$\forall x \exists y \exists z \neg ((P(y) \supset Q(z)) \supset (P(x) \supset Q(x)))$$

Convert to clausal form

$$\forall x \exists y \exists z \neg ((\neg P(y) \vee Q(z)) \vee (\neg P(x) \vee Q(x)))$$

$$\equiv \forall x \exists y \exists z ((\neg P(y) \vee Q(z)) \wedge (P(x) \wedge \neg Q(x)))$$

skolemize to

$$\forall x ((\neg P(f(x)) \vee Q(g(x))) \wedge P(x) \wedge \neg Q(x))$$

Resolution  
Proof

1  $P(y)$

2  $\neg Q(z)$

3  $\neg P(f(x)) \vee Q(g(x))$

4  $Q(g(x))$

5  $\square$

conversion to  
clausal form  
yields 3 clauses

$$R[1, 3] \theta = \{y = f(x)\}$$

$$R[2, 4] \theta = \{z = g(x)\}$$

3a) It is satisfiable.

Let  $\mathcal{I}$  be the interpretation  $\langle \Delta_{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$  with

$$\Delta_{\mathcal{I}} = \{d\}, \quad A^{\mathcal{I}} = B^{\mathcal{I}} = C^{\mathcal{I}} = D^{\mathcal{I}} = E^{\mathcal{I}} = r^{\mathcal{I}} = \emptyset$$

Clearly  $\mathcal{I} \models C \sqsubseteq B$  since  $C^{\mathcal{I}} \subseteq B^{\mathcal{I}}$  as  $C^{\mathcal{I}} = \emptyset$ .

Similarly,  $\mathcal{I}$  satisfies the other 4 axioms in  $T$ .

b)  $D$  is satisfiable wrt  $T$ .

Let  $\mathcal{I}$  be the interpretation  $\langle \Delta_{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$  with

$$\Delta_{\mathcal{I}} = \{d\}, \quad A^{\mathcal{I}} = B^{\mathcal{I}} = C^{\mathcal{I}} = E^{\mathcal{I}} = r^{\mathcal{I}} = \emptyset \text{ and } D^{\mathcal{I}} = \{d\}.$$

Clearly the first 4 axioms are satisfied because the extension of the left hand side is empty so the  $\sqsubseteq$  is satisfied.

$$\mathcal{I} \models D \sqsubseteq \neg C \text{ since } D^{\mathcal{I}} \subseteq (\neg C)^{\mathcal{I}} \\ \text{as } \{d\} \subseteq \Delta_{\mathcal{I}} \setminus \emptyset.$$

also  $D^{\mathcal{I}}$  is non empty

c)  $C \sqcap D$  is not satisfiable wrt  $T$ .

Since  $T$  contains  $D \sqsubseteq \neg C$  we must have

$$D^{\mathcal{I}} \subseteq \Delta_{\mathcal{I}} \setminus C^{\mathcal{I}} \text{ and thus } D^{\mathcal{I}} \cap C^{\mathcal{I}} = \emptyset$$

Any interpretation  $\mathcal{I}$  that satisfies  $T$  cannot make  $(C \sqcap D)^{\mathcal{I}}$  non empty.

4 a)

$$x_0 \cdot \{C_0\} \text{ where } C_0 = (\forall R. \forall R. \forall R. \forall R. \neg A) \wedge = C_1$$

$$(\forall R. \exists R. \forall R. \exists R. B) \wedge = C_2$$

$$(\exists R. \forall R. \exists R. \exists R. C) \wedge = C_3$$

$$(\forall R. \forall R. \forall R. \exists R. A) = C_4$$

→ 4 times  
 $\wedge$

$$x_0 \cdot \{C_0, C_1, C_2, C_3, C_4\}$$

→  
 $\exists$   $x_0$  as above

$$\begin{array}{c} R \\ | \\ x_1 \cdot \{ \forall R. \exists R. \exists R. C \} \end{array}$$

→ 3 times  
 $\forall$

$$x_0 \cdot \text{as above}$$

$$\begin{array}{c} R \\ | \\ x_1 \cdot \{ \forall R. \exists R. \exists R. C, \forall R. \forall R. \forall R. \neg A, \\ \exists R. \forall R. \exists R. B, \forall R. \forall R. \exists R. A \} \end{array}$$

→  
 $\exists$

$$x_0 \cdot \text{as above}$$

$$\begin{array}{c} R \\ | \\ x_1 \cdot \{ C_5, C_6, C_7, C_8 \} \\ | \\ x_2 \cdot \{ \forall R. \exists R. B \} \end{array}$$

→ 3 times  
 $\forall$

$$x_0 \cdot \text{as above}$$

$$\begin{array}{c} R \\ | \\ x_1 \cdot \text{as above} \\ | \\ x_2 \cdot \{ \forall R. \exists R. B, \exists R. \exists R. C, \forall R. \forall R. \neg A, \\ \forall R. \exists R. A \} \end{array}$$

$\rightarrow \exists$   
 $x_0 \cdot \text{as above}$   
 $| R$   
 $x_1 \cdot \text{as above}$   
 $| R$   
 $x_2 \cdot \text{as above}$   
 $| R$   
 $x_3 \cdot \{ \exists R C \}$

$\rightarrow \exists$  times  
 $\forall$

$x_0 \cdot \text{as above}$   
 $| R$   
 $x_1 \cdot \text{as above}$   
 $| R$   
 $x_2 \cdot \text{as above}$   
 $| R$   
 $x_3 \cdot \{ \exists R C, \exists R B, \forall R \neg A, \exists R A \}$

$\rightarrow \exists$

$x_0 \cdot \text{as above}$   
 $| R$   
 $x_1 \cdot \text{as above}$   
 $| R$   
 $x_2 \cdot \text{as above}$   
 $| R$   
 $x_3 \cdot \text{as above}$   
 $| R$   
 $x_4 \cdot \{ A \}$

$\rightarrow \forall$   
 $x_0 \cdot \text{as before}$   
 $| R$   
 $x_1 \cdot \text{as before}$   
 $| R$   
 $x_2 \cdot \text{as before}$   
 $| R$   
 $x_3 \cdot \text{as before}$   
 $| R$   
 $x_4 \cdot \{ A, \neg A \}$   
 $X \text{ clash}$

unsatisfiable

4b) call the concept  $C_0$

$$X_0 = \{C_0\}$$

→ 3 times

$$X_0 = \{C_0, \forall R \exists R (\overset{C_1}{\forall R A \sqcup \forall R B \sqcup \forall R C}), \forall R \forall R (\overset{C_2}{\exists R \neg A \sqcup \exists R \neg B}), \forall R \forall R \overset{C_3}{\exists R \neg C}, \exists R.C\}$$

$$\rightarrow X_0 = \{C_0, C_1, C_2, C_3, \exists R.C\}$$

$$\begin{array}{c} X_0 \\ \downarrow R \\ X_1 \end{array} \quad \{C\}$$

→  $X_0$  as above

$$\begin{array}{c} X_0 \\ \downarrow R \\ X_1 \end{array} \quad \{C, \exists R (\overset{C_4}{\forall R A \sqcup \forall R B \sqcup \forall R C}), \forall R (\overset{C_5}{\exists R \neg A \sqcup \exists R \neg B}), \forall R \overset{C_6}{\exists R \neg C}\}$$

→  $X_0$  as above

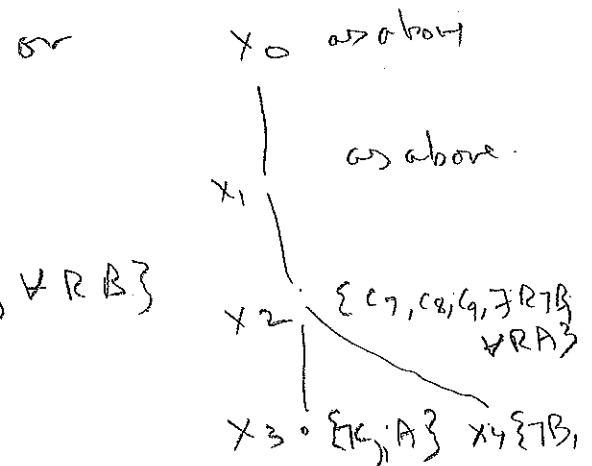
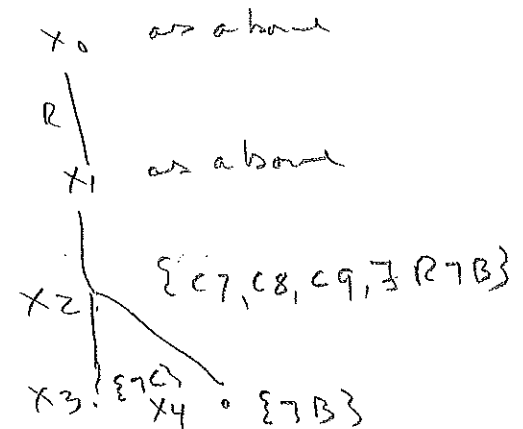
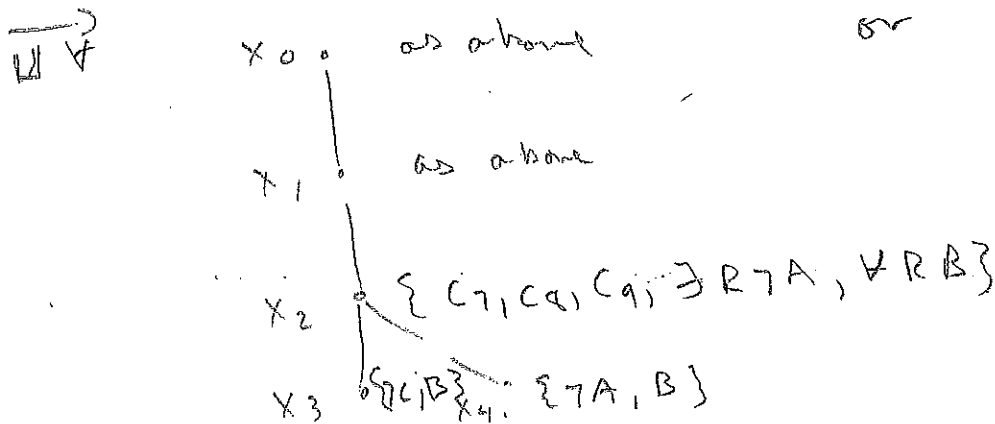
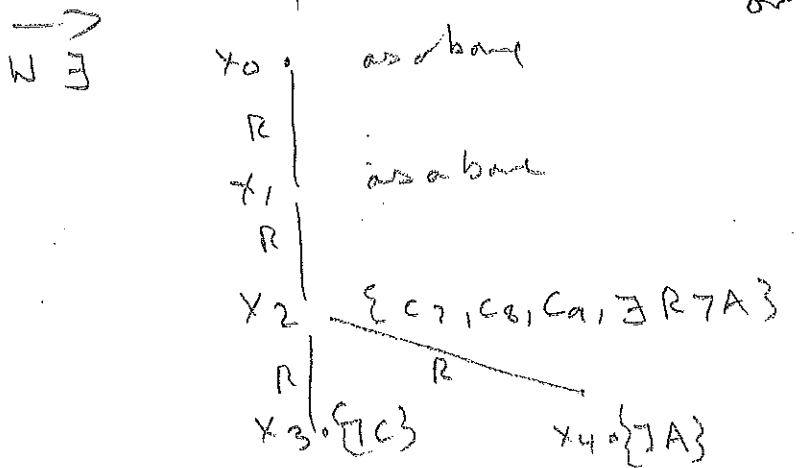
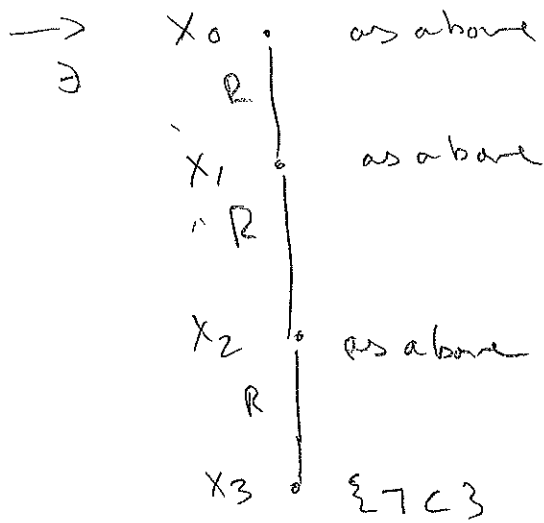
$$\begin{array}{c} X_0 \\ \downarrow R \\ X_1 \\ \downarrow R \\ X_2 \end{array} \quad \begin{array}{l} \{C, C_4, C_5, C_6\} \\ \{\forall R A \sqcup \forall R B \sqcup \forall R C\} \end{array}$$

→ twice

$X_0$  as above

$X_1$  as above

$$\begin{array}{c} X_0 \\ \downarrow R \\ X_1 \\ \downarrow R \\ X_2 \end{array} \quad \{ \overset{C_7}{\forall R A \sqcup \forall R B \sqcup \forall R C}, \overset{C_8}{\exists R \neg A \sqcup \exists R \neg B}, \overset{C_9}{\exists R \neg C} \}$$



✓

$$D = \{ x_0, x_1, x_2, x_3, x_4 \}$$

$$I(R) = \{ \langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \langle x_2, x_4 \rangle \}$$

$$I(A) = \{ \}$$

$$I(B) = \{ x_3, x_4 \}$$

$$I(C) = \{ \}$$

Same except

✓

$$I(A) = \{ x_3, x_4 \}$$

$$I(B) = \{ \}$$