EECS-1019C: ASSIGNMENT #8

Out of 30 points.

Section 5.1 [15pt]

20. [5pt] Prove that

$$\sum_{j=0}^{n} (-\frac{1}{2})^j = \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n}$$

Let $S(n)$ be equal to the summation above (the lefthand side) and let $f(n)$ equal the formula above (the righthand side).

basis case.

For $n = 0$: $S(0) = 1$ and $f(0) = 1$.

inductive hypothesis.

Assume for some $k \geq 0$ that $S(k) = f(k)$.

inductive step.

$$\begin{aligned}
S(k+1) &= S(k) + (-\tfrac{1}{2})^{k+1} && \text{by inductive hypothesis} \\
&= \tfrac{2^{k+1}+(-1)^k}{3\cdot 2^k} + (-\tfrac{1}{2})^{k+1} \\
&= \tfrac{2}{3} + \tfrac{(-1)^k}{3\cdot 2^k} + (-\tfrac{1}{2})^{k+1} \\
&= \tfrac{2}{3} - (-1)^{k+1}\tfrac{2}{3\cdot 2^{k+1}} + (-1)^{k+1}\tfrac{3}{3\cdot 2^{k+1}} \\
&= \tfrac{2}{3} + (-1)^{k+1}\tfrac{1}{3\cdot 2^{k+1}} \\
&= \tfrac{2^{k+2}+(-1)^{k+1}}{3\cdot 2^{k+1}} \\
&= f(k+1)
\end{aligned}$$

22. [5pt] For which nonnegative integers n is $n^2 \leq n!$? Prove your answer.

$1^2 = 1 \leq 1! = 1$
$2^2 = 4 > 2! = 2$
$3^2 = 9 > 3! = 6$
$4^2 = 16 < 4! = 24$

basis case.

$n = 4$: $4^2 = 16 < 4! = 24$

inductive hypothesis.

Assume for some k for $k \geq 4$ that $k^2 < k!$.

inductive step.

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1\\ &< k^2 + 3k \quad \textit{since } k > 1\\ &< 2k^2 \quad \textit{since } k > 3\\ &< 2(k!) \quad \textit{by inductive hypothesis}\\ &< (k+1)k! \quad \textit{since } k+1 > 2\\ &= (k+1)!\end{aligned}$$

32. [5pt] Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

basis case.

$n = 0$: 3 divides evenly into 0.

inductive hypothesis.

Assume for some k for $k \geq 1$ that 3 evenly divides into $k^3 + 2k$.

inductive step.

$$\begin{aligned}(k+1)^3 + 2(k+1) &= (k^3 + 3k^2 + 3k + 1) + (2k + 2)\\ &= k^3 + 3k^2 + 5k + 3\\ &= (k^3 + 2k) + 3(k^2 + k + 1)\end{aligned}$$

$(k^3 + 2k)$ is evenly divisible by 3 by the inductive hypothsis. $3(k^2 + k + 1)$ is clearly evenly divisible by 3.

Section 5.2 [5pt]

12. [5pt] Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on.

[Hint: For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

basis case.

$n = 1$: 1 is the sum of $\{2^0\}$.

inductive hypothesis.

Assume for some k for $k \geq 1$, for $1 \leq i \leq k$, i is the sum of distinct powers of two.

inductive step.

Case that $k + 1$ is odd.

k is the sum of distinct powers of two, by the induction hypothesis. But 2^0 is not one of the powers of two in this sum as k is even. Add 2^0 to k's set of powers of two (that sum to it): this new set of powers of two sums to $k + 1$.

Case that $k + 1$ is even.

$(k + 1)/2$ is an integer. By the induction hypothesis, there is a set T of powers of two that, summed, equals $(k + 1)/2$: Note that $\sum_{t \in T} 2t = k + 1$. Thus, the set $\{2t \mid t \in T\}$ represents distinct powers of two that sum to $k + 1$.

Section 5.3 [10pt]

12. [5pt] f_n is the nth Fibonacci number.

Prove that $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

Note that $f_0 = 0$, $f_1 = 1$, *and* $f_2 = 1$. *Let* $S(n) = f_1^2 + f_2^2 + \ldots + f_n^2$ *and* $g(n) = f_n f_{n+1}$.

basis case.

$n = 1$: $S(1) = f_1^2 = 1^2 = 1$. $g(1) = f_1 f_2 = 1 \cdot 1 = 1$.

inductive hypothesis.

Assume for some k *for* $k \geq 1$ *that* $S(k) = g(k)$.

inductive step.

$$\begin{aligned}
S(k+1) &= S(k) + f_{k+1}^2 \\
&= f_k f_{k+1} + f_{k+1}^2 && \textit{by induction hypothesis} \\
g(k+1) &= f_{k+1} f_{k+2} \\
&= f_{k+1}(f_{k+1} + f_k) && \textit{by definition of Fibonnacci} \\
&= f_k f_{k+1} + f_{k+1}^2
\end{aligned}$$

14. [5pt] f_n is the *nth* Fibonacci number.

Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ when n is a positive integer.

Note that $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, and $f_3 = 2$. Let $g(n) = f_{n+1}f_{n-1} - f_n^2$ and $h(n) = (-1)^n$.
Proof by strong induction.
basis cases.

$n = 1$: $g(1) = f_2 f_0 - f_1^2 = 1 \cdot 0 - 1^2 = -1$. $h(1) = (-1)^1 = -1$.
$n = 2$: $g(2) = f_3 f_1 - f_2^2 = 2 \cdot 1 - 1^2 = 1$. $h(2) = (-1)^2 = 1$.

inductive hypothesis.

Assume for some k for $k \geq 1$, for $1 \leq i \leq k$, $g(i) = h(i)$.

inductive step.

$$
\begin{aligned}
g(k+1) &= f_{k+2}f_k - f_{k+1}^2 \\
&= (f_{k+1} + f_k)(f_{k-1} + f_{k-2}) - (f_k + f_{k-1})^2 && \textit{by definition of Fibonnacci} \\
&= (f_{k+1}f_{k-1} + f_{k+1}f_{k-2} + f_k f_{k-1} + f_k f_{k-2}) - (f_k^2 + 2f_k f_{k-1} + f_{k-1}^2) \\
&= (f_{k+1}f_{k-1} - f_k^2) + (f_k f_{k-2} - f_{k-1}^2) + f_{k+1}f_{k-2} - f_k f_{k-1} \\
&= (-1)^k + (-1)^{k-1} + f_{k+1}f_{k-2} - f_k f_{k-1} && \textit{by induction hypothesis} \\
&= f_{k+1}f_{k-2} - f_k f_{k-1} \\
&= (f_k + f_{k-1})f_{k-2} - (f_{k-1}f_{k-2})f_{k-1} && \textit{by definition of Fibonnacci} \\
&= f_k f_{k-2} + f_{k-1}f_{k-2} - f_{k-1}^2 - f_{k-1}f_{k-2} \\
&= f_k f_{k-2} - f_{k-1}^2 \\
&= (-1)^{k-1} && \textit{by induction hypothesis} \\
&= (-1)^{k+1} \\
&= h(k+1)
\end{aligned}
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