Homework Assignment #2
Due: September 25, 2014 at 4:00 p.m.

1. Instead of binary or decimal, the Kingdom of Leutonia uses an unusual system to represent numbers, based on the Fibonacci sequence. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined recursively as follows.

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \end{aligned}$$

A Leutonian number is a string of 0's and 1's that begins with a 1 and never has two consecutive 1's. If $s = s_\ell s_{\ell-1} \ldots s_1$ is such a string of length ℓ, where each s_i is in $\{0, 1\}$, the number represented by s is is $n(s) = \sum_{i=1}^{\ell} s_i \cdot F_i$.

For example, $n(1000101) = F_7 + F_3 + F_1 = 21 + 3 + 1 = 25$.

(a) Write out the Leutonian numbers that represent the first 12 positive integers.

(b) Prove: For every $\ell \geq 1$, if s is a Leutonian number of length ℓ, $n(s) \geq F_\ell$.

(c) Prove: For every $\ell \geq 1$, if s is a Leutonian number of length ℓ, $n(s) < F_{\ell+1}$.

(d) Prove: For every $\ell \geq 1$, every number in $S_\ell = \{x \in \mathbb{N} : 1 \leq x < F_{\ell+1}\}$ can be represented by a Leutonian number of length at most ℓ.

(Remark: this representation is actually unique, but you do not have to prove that.)

(e) Draw the transition diagram of a deterministic finite automaton that accepts an input string if and only if it is a Leutonian number. You need not prove your answer is correct.