CSE4221: Assignment 1

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1. Find the homogeneous transformation T_1^0 where:

   (a) $\{1\}$ has the same orientation as $\{0\}$ and the origin of $\{1\}$ is translated relative to the origin of $\{0\}$ by $d_1^0 = \begin{bmatrix}1 & -1 & 2\end{bmatrix}^T$.

   (b) The origin of $\{1\}$ is coincident with the origin of $\{0\}$, and $\hat{x}_1^0 = \hat{y}_0^0$, $\hat{y}_1^0 = -\hat{z}_0^0$, and $\hat{z}_1^0 = -\hat{x}_0^0$.

   (c) The origin of $\{0\}$ is translated relative to the origin of $\{1\}$ by $d_0^1 = \begin{bmatrix}0 & 0 & -1.7321\end{bmatrix}^T$, and
   $\hat{x}_1^0 = \begin{bmatrix}0.7887 & -0.2113 & -0.5774\end{bmatrix}^T$,
   $\hat{y}_1^0 = \begin{bmatrix}-0.2113 & 0.7887 & -0.5774\end{bmatrix}^T$, and
   $\hat{z}_1^0 = \begin{bmatrix}0.5774 & 0.5774 & 0.5774\end{bmatrix}^T$.

2. From the lectures you know that the columns of the rotation matrix R_1^0 are just the x, y, and z axes of frame 1 expressed in frame 0. What are the rows of R_1^0?

3. Find the missing elements of the following rotation matrices. Show your work, or explain your reasoning. It may be the case that there is no unique solution, in which case you should find all possible solutions. *Hint: Consider using the cross product.*

   (a) $\begin{bmatrix} \cdot & 0 & -1 \\ \cdot & 0 & 0 \\ \cdot & -1 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} \sqrt{2}/2 & \cdot & 0 \\ \cdot & 0 & 1 \\ \cdot & -\sqrt{2}/2 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} \sqrt{2}/2 & 0 & \cdot \\ \sqrt{2}/2 & \cdot & \cdot \\ \cdot & \cdot & 0 \end{bmatrix}$

4. Consider the following 4×4 homogeneous transformation matrices:

$$\begin{aligned}
R_{x,a} &: \text{rotation about } x \text{ by an angle } a \\
R_{y,a} &: \text{rotation about } y \text{ by an angle } a \\
R_{z,a} &: \text{rotation about } z \text{ by an angle } a \\
D_{x,a} &: \text{translation along } x \text{ by a distance } a \\
D_{y,a} &: \text{translation along } y \text{ by a distance } a \\
D_{z,a} &: \text{translation along } z \text{ by a distance } a
\end{aligned}$$

Write the matrix product giving the overall transformation for the following sequences (do not perform the actual matrix multiplications):

(a) The following rotations all occur in the moving frame.

  i. Rotate about the current z-axis by angle ϕ.
  ii. Rotate about the current y-axis by angle θ.
  iii. Rotate about the current z-axis by angle ψ.

  Note: This yields the ZYZ-Euler angle rotation matrix.

(b) The following rotations all occur in a fixed (world) frame.

  i. Rotate about the world x-axis by angle ψ.
  ii. Rotate about the world y-axis by angle θ.
  iii. Rotate about the world z-axis by angle ϕ.

  Note: This yields the roll, pitch, yaw (RPY) rotation matrix.

(c) The following transformations all occur in the moving frame.

  i. Rotate about the current z-axis by angle θ_i.
  ii. Translate along the current z-axis by a distance d_i.
  iii. Translate along the current x-axis by a distance a_i.
  iv. Rotate about the current x-axis by angle α_i.

  Note: This is the Denavit-Hartenberg transformation matrix.

5. Give a 3×3 rotation matrix R prove that the angle of rotation can be computed as

$$\theta = \arccos((\text{trace}(R) - 1)/2)$$

where the trace of a matrix is defined as the sum of its diagonal elements. Assume that R has the form of a rotation of θ degrees a unit axis $\hat{k}$ (as shown in the slides from Jan 16).
Note: This fact turns out to be very useful when you need to extract the rotation angle from a given rotation matrix.

6. Prove that the length of a vector is unchanged by rotation; that is, prove that $\|Rv\|^2 = \|v\|^2$ for every (3D) rotation matrix R where $\|\cdot\|$ denotes the length of a vector.
Hint: $\|x\|^2 = x \cdot x$ for any vector x.

7. Suppose that you have two coordinate frames $\{0\}$ and $\{1\}$ and you know T_1^0 the pose of frame $\{1\}$ relative to $\{0\}$. Suppose that you have some rigid transformation U defined in frame $\{1\}$; what is U expressed in frame $\{0\}$?
Hint: Construct some simple examples of T_1^0 and U to test your answer; you can test more complicated examples using Matlab.

8. Not surprisingly, rigid transformations are used in many different application areas: robotics, computer graphics, augmented and virtual reality, and computer vision all make heavy use of rigid transformations.

   Computer-aided surgery is another application area where rigid transformations are widely used. For example, treatment of bony deformity often requires cutting the deformed bone, followed by re-alignment of the bone fragments, followed by fixation. In computer-aided interventions, the re-alignment is often planned virtually using models derived from pre-operative medical images (see

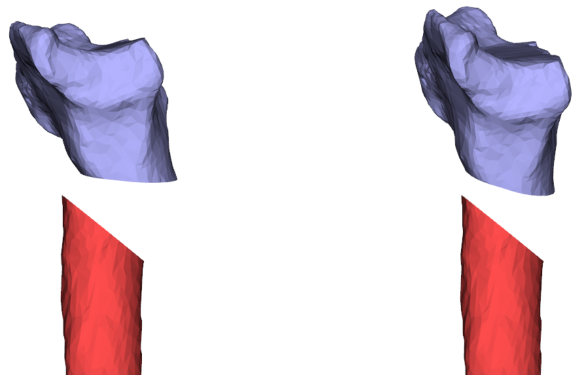

Figure 1: Left: The planned correction for a distal radial osteotomy. The transformation of the blue distal fragment relative to the red proximal fragment is given as $T^p_{d,\text{plan}}$. Right: The actual correction achieved. The transformation of the blue distal fragment relative to the red proximal fragment is measured to be $T^p_{d,\text{actual}}$.

Figure 1 left). To assess the accuracy of a performed procedure, it is necessary to compare the actual re-alignment achieved (see Figure 1 right) to the planned correction.

With reference to Figure 1, find an expression for the correction error Δ in terms of $T^p_{d,\text{plan}}$ and $T^p_{d,\text{actual}}$. Note that Δ is a homogeneous transformation matrix.

Hint: Suppose that $T^p_{d,\text{plan}} = T_{z,-5}$ (a rotation of $-5°$ about z) and that $T^p_{d,\text{actual}} = T_{z,10}$ (a rotation of $10°$ about z); then the correction error is $\Delta = T_{z,15}$.