CSE6338: Assignment 1

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1. Find the homogeneous transformation T_1^0 where:

   (a) $\{1\}$ has the same orientation as $\{0\}$ and the origin of $\{1\}$ is translated relative to the origin of $\{0\}$ by $d_1^0 = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$.

   (b) The origin of $\{1\}$ is coincident with the origin of $\{0\}$, and $\hat{x}_1^0 = \hat{y}_0^0$, $\hat{y}_1^0 = -\hat{z}_0^0$, and $\hat{z}_1^0 = -\hat{x}_0^0$.

   (c) The origin of $\{0\}$ is translated relative to the origin of $\{1\}$ by $d_0^1 = \begin{bmatrix} 0 & 0 & -1.7321 \end{bmatrix}^T$, and $\hat{x}_1^0 = \begin{bmatrix} 0.7887 & -0.2113 & -0.5774 \end{bmatrix}^T$, $\hat{y}_1^0 = \begin{bmatrix} -0.2113 & 0.7887 & -0.5774 \end{bmatrix}^T$, and $\hat{z}_1^0 = \begin{bmatrix} 0.5774 & 0.5774 & 0.5774 \end{bmatrix}^T$.

2. Consider the following 4×4 homogeneous transformation matrices:

$$
\begin{aligned}
R_{x,a} &: \text{rotation about } x \text{ by an angle } a \\
R_{y,a} &: \text{rotation about } y \text{ by an angle } a \\
R_{z,a} &: \text{rotation about } z \text{ by an angle } a \\
D_{x,a} &: \text{translation along } x \text{ by a distance } a \\
D_{y,a} &: \text{translation along } y \text{ by a distance } a \\
D_{z,a} &: \text{translation along } z \text{ by a distance } a
\end{aligned}
$$

   Write the matrix product giving the overall transformation for the following sequences (do not perform the actual matrix multiplications):

   (a) The following rotations all occur in the moving frame.

      i. Rotate about the current z-axis by angle ϕ.
      ii. Rotate about the current y-axis by angle θ.
      iii. Rotate about the current z-axis by angle ψ.

      Note: This yields the ZYZ-Euler angle rotation matrix.

   (b) The following rotations all occur in a fixed (world) frame.

      i. Rotate about the world x-axis by angle ψ.
      ii. Rotate about the world y-axis by angle θ.
      iii. Rotate about the world z-axis by angle ϕ.

      Note: This yields the roll, pitch, yaw (RPY) rotation matrix.

3. Give a 3×3 rotation matrix R prove that the angle of rotation can be computed as

$$\theta = \arccos((\text{trace}(R) - 1)/2)$$

where the trace of a matrix is defined as the sum of its diagonal elements. You might consider starting by assuming R has the form of a rotation of θ degrees a unit axis $\hat{k}$.

4. A quaternion is another representation of rotation in 3D. The quaternion $Q = (q_w, q_x, q_y, q_z)$ can be thought of as being a scalar q_w and a vector $\vec{q} = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^T$. Given two quaternions $A = (a_w, \vec{a})$ and $B = (b_w, \vec{b})$, the quaternion product $C = AB$ is defined as

$$\begin{aligned} c_w &= a_w b_w - \vec{a} \cdot \vec{b} \\ \vec{c} &= a_w \vec{b} + b_w \vec{a} + \vec{a} \times \vec{b} \end{aligned}$$

where $\vec{a} \times \vec{b}$ is the cross product of $\vec{a}$ and $\vec{b}$.

(a) Show that $Q_I Q = Q Q_I = Q$ for every unit quaternion Q where $Q_I = (1, 0, 0, 0)$, i.e., Q_I is the identity quaternion.

(b) The conjugate Q^* of a quaternion $Q = (q_w, \vec{q})$ is given by $Q^* = (q_w, -\vec{q})$. Show that $Q^* Q = Q Q^* = (1, 0, 0, 0)$, i.e., Q^* is the inverse of Q.

(c) The quaternion $Q = (q_w, \vec{q})$ where $q_w = \cos\frac{\theta}{2}$ and $\vec{q} = \begin{bmatrix} k_x \sin\frac{\theta}{2} & k_y \sin\frac{\theta}{2} & k_z \sin\frac{\theta}{2} \end{bmatrix}^T$ represents the rotation of angle θ about the unit vector $\hat{k} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$. A vector $\vec{p} = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T$ can be rotated using the quaternion product QPQ^* where P is the quaternion $(0, \vec{p})$. Show that this is true for a rotation of angle θ about the z-axis.

5. Treatment of bony deformity often requires cutting the deformed bone, followed by re-alignment of the bone fragments, followed by fixation. In computer-aided interventions, the re-alignment is often planned virtually using models derived from pre-operative medical images. To assess the accuracy of a performed procedure, it is necessary to compare the actual re-alignment achieved to the planned correction.

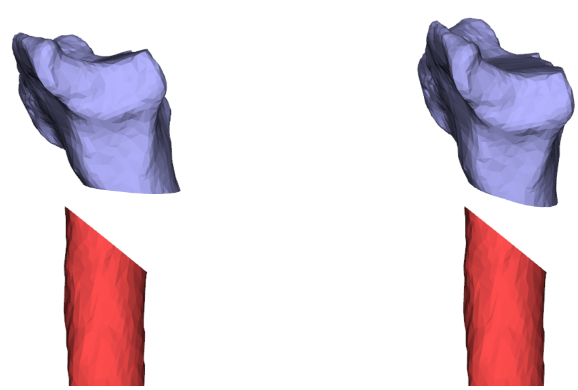

Figure 1: Left: The planned correction for a distal radial osteotomy. The transformation of the blue distal fragment relative to the red proximal fragment is given as $T^p_{d,\text{plan}}$. Right: The actual correction achieved. The transformation of the blue distal fragment relative to the red proximal fragment is measured to be $T^p_{d,\text{actual}}$.

Find an expression for the correction error Δ in terms of $T^p_{d,\text{plan}}$ and $T^p_{d,\text{actual}}$. Note that Δ is a homogeneous transformation matrix.

Hint: Suppose that $T^p_{d,\text{plan}} = T_{z,-5}$ (a rotation of $-5°$ about z) and that $T^p_{d,\text{actual}} = T_{z,10}$ (a rotation of $10°$ about z); then the correction error is $\Delta = T_{z,15}$.

6. Reconsider the situation in Question 5. The planned correction is usually expressed in the coordinate frame of the medical image; in the example given in Question 5 this frame would be the CT coordinate frame $\{\text{CT}\}$. The actual correction would be measured in some other coordinate frame. If we were performing a laboratory study using plastic bones then the actual correction might be measured by using a 3D laser scanner to scan the physical plastic specimen; thus the actual correction might be expressed in the laser scanner frame $\{\text{LS}\}$.

   Given $T^{\text{CT}}_{\text{LS}}$ and the actual correction T_{actual} expressed in $\{\text{LS}\}$, find an expression for the actual correction expressed in $\{\text{CT}\}$.

7. If you have never used Matlab, then consider working through Chapters 1 and 2 of the *Getting Starting Guide*:

   `http://www.mathworks.com/help/pdf_doc/matlab/getstart.pdf`

   and read the document *Working with Functions in Files*

   `http://www.mathworks.com/help/techdoc/matlab_prog/f7-41453.html`

   before attempting this question.

   If you want to install Matlab on your own computer then follow the instructions here:

   `http://www.cse.yorku.ca/tdb/_doc.php/userg/sw/matlab`

   Implement a RANSAC version of Horn's absolute orientation method as a Matlab function. Your implementation should allow the caller to specify the error tolerance of the model, the threshold size of the consensus set, and the maximum number of trials to attempt.

   Use the function `makepoints.m` available from the assignment web page to test your function.

   Write a simulation that runs 10000 trials of your solution using a point set of size 10 with 4 random outliers. Use the function `makepoints.m` available from the assignment web page to generate the point sets to be registered each time through the loop (the function produces a noisy point set L and a clean point set R; you want to register L to R). For each trial compute the rotational error and translational error of the estimated registration transformation (for the point sets generated using `makepoints.m` the true registration transformation is always the identity transformation). Plot a histogram of your results using the `hist` function. For example, assuming rotation errors stored in a Matlab variable Er, the histogram of errors I obtain using regular Horn's method can be plotted using:

```
hist(Er, 20);
xlabel('rotation error in degrees')
ylabel('frequency')
```

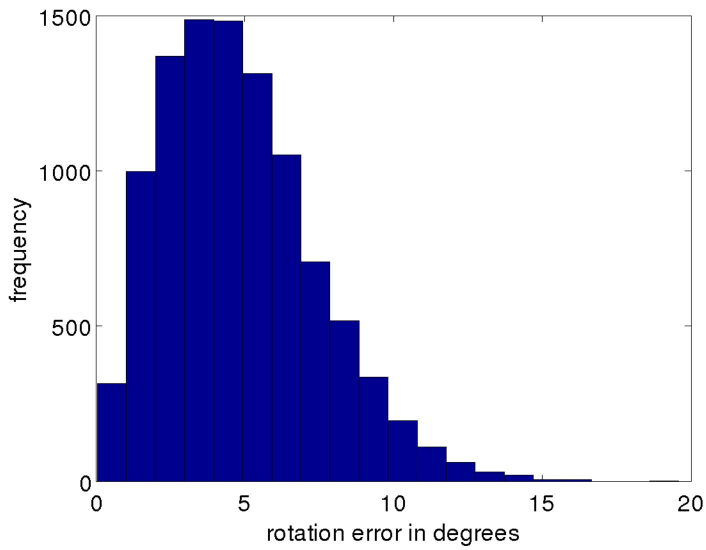

Figure 2: Histogram (using 20 bins) of rotation errors in degrees using Horn's method.

Submit your solution by emailing it to me.