Math/CSE 1019C: Discrete Mathematics for Computer Science
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Course page:
http://www.cse.yorku.ca/course/1019

- Last date to drop courses without receiving a grade : Nov 9th
- No Assignment is released today!
- Test 2 on Nov 5th,
  - Ch2.1-2.5
  - 7pm-8:20pm
  - Location: SLH F (Last name from A-L) and SLH A (Last name from M-Z)
  - Lecture: 8:40pm, SLH A.
- You may pick up your Assignment 4 during my office hour: Nov 5th 2:00pm – 4:00pm (TEL 3056)

Mathematical Induction

- Proof methods: direct proof, proof by cases, proof by contraposition, proof by contradiction, disproof by counterexample
- Mathematical induction
  - very simple and powerful proof technique
  - often used to prove P(x) is true for all positive integers
  - "Guess" and verify strategy

Principle of Mathematical Induction

$(P(1) \wedge \forall k(P(k) \rightarrow P(k+1)) \rightarrow \forall n P(n)$

- To prove that $\forall n P(n)$, where $n \in Z^+$ and P(n) is a propositional function, we complete two steps:
  - i) Basis step: Verify P(1) is true
  - ii)Inductive step: Show $P(k) \rightarrow P(k+1)$ is true for arbitrary $k \in Z^+$

Mathematical Induction

- Knowing it is true for the first element means it must be true for the next element, i.e. the second element
- Knowing it is true for the second element implies it is true for the third and so forth.

$$\frac{P(1) \quad P(k) \rightarrow P(k+1)}{P(2)}, \quad \frac{P(2) \quad P(k) \rightarrow P(k+1)}{P(3)}, \quad \ldots$$

- Need a starting point (Base case)

- How to show P(1) is true?
  - Replace n by 1 in P(n)
- How to show $P(k) \rightarrow P(k+1)$ is true?
  - Direct proof is normally used
  - (Inductive Hypothesis) Assume P(k) is true for some arbitrary k
  - Then show P(k+1) is true

Proving Summation

- Example: Show that $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$, where $n \in Z^+$
- Proof:
  - Basis case P(1): LHS=1^2 = 1, RHS=1*(1+1)(2*1+1)/6=1
  - Inductive step:
    - Assume P(k) is true:
      $1^2 + 2^2 + 3^2 + \ldots + k^2 = k(k+1)(2k+1)/6$
    - For P(k+1):
      Show that $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = (k+1)(k+2)(2k+3)/6$
      (Details on Board)
    - We showed P(k+1) is true under the assumption that P(k) is true.
  - By mathematical induction P(n) is true for all positive integers

Proving Inequalities

- Example: $n < 4^n$, where $n \in Z^+$
- Proof
  - Basis case: P(1) holds since 1<4
  - Inductive step:
    - Assume P(k) is true: $k < 4^k$
    - For P(k+1):
      $k+1 < 4^k + 1 < 4^k + 4^k = 2*4^k < 4*4^k = 4^{k+1}$
  - By mathematical induction P(n) is true for all positive integers

Proving divisibility

- Example: Prove that n^3-n is divisible by 3 whenever n is a positive integer.
  - Basis step: P(1): $1^3-1=0$, which is divisible by 3.
  - Inductive step:
    - Assume P(k) is true: $k^3 - k$ is divisible by 3
    - For P(k+1):
      $(k+1)^3 - (k+1)$
      $= k^3 + 3k^2 + 3k + 1 - k - 1$
      $= (k^3 - k) + 3(k^2 + k)$
      From the assumption, k^3-k is divisible by 3, so P(k+1) is true.
  By mathematical induction, P(n) is true for all positive integers

Mathematical Induction (Base case n≠1)

- Basis case doe not have to be n=1
- How to show that P(n) is true for n=b, b+1, b+2,.... where b is an integer other than 1?
  - i) Basis step: Verify P(b) is true
  - ii)Inductive step: Show P(k)→P(k+1) is true for arbitrary $k \in Z$

- Show that $1+2+2^2+\ldots+2^n = 2^{n+1} - 1$, where $n \in N$
- Proof by induction:
  - Basis step: P(0): LHS=1, RHS = $2^1 - 1$ = 1.
  - Inductive step:
    - Assume P(k) is true for arbitrary $k \in N$,
      $1+2+\ldots+2^k = 2^{k+1} - 1$
    - Need to show P(k+1): $1+2+\ldots+2^k+2^{k+1} = 2^{k+2} - 1$ is true.
      LHS = $(1+2+\ldots+2^k)+2^{k+1} = 2^{k+1}-1+2^{k+1} = 2^{k+2}-1$
      So LHS = RHS. We showed P(k+1) is true under the assumption that P(k) is true.
  - By mathematical induction P(n) is true for all natural numbers

- Prove $\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1}$ if $r \neq 1$
- Proof by induction:
  - P(n): $a+ar+ar^2\ldots + ar^n = (ar^{n+1} - a)/(r-1)$
  - Basis case: P(0): a = (ar−a)/(r−1). So, P(0) is true
  - Inductive step:
    - Assume P(k) is true for arbitrary $k \in N$,
      $a+ar+ar^2\ldots + ar^k = (ar^{k+1} - a)/(r-1)$
    - Need to show P(k+1):
      $a+ar+ar^2\ldots + ar^k + ar^{k+1} = (ar^{k+2} - a)/(r-1)$ is true.
    - LHS = $(ar^{k+1}-a)/(r-1) + ar^{k+1}$
      $= (ar^{k+1} - a + ar^{k+2} - ar^{k+1})/(r-1)$
    - So LHS = RHS.
    - We showed P(k+1) is true under the assumption that P(k) is true.
  - By mathematical induction P(n) is true for all natural numbers

A wrong proof, WHY?

- Show that all horses are the same color.
- Proof:
  - We do induction on the size of sets of horses of the same color.
  - Basis step: Obviously all sets of 0 horses (and all sets with 1 horse) are the same color
  - Inductive step:
    - Assume all horses are the same color for k horses
    - Now show it must be true for all sets of k+1 horses
    - Every set of k+1 horses has an overlap of horses which are the same color.
    - So k+1 horses have the same color.
    - Therefore all horses have the same color.

Strong Induction

$(P(1) \wedge \forall k(P(1) \wedge P(2) \wedge \ldots \wedge P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n)$

- To prove that $\forall n P(n)$, where $n \in Z^+$ and P(n) is a propositional function, we complete two steps:
  - i) Basis step: Verify P(1) is true
  - ii)Inductive step: Show $P(1) \wedge P(2) \wedge \ldots \wedge P(k) \rightarrow P(k+1)$ is true for arbitrary $k \in Z^+$

Strong Induction Variation

- A more general strong induction can handle cases where the inductive step is valid only for integers greater than a particular integer.

- To prove that P(n) is true for all integer $n \geq b$, we complete two steps:
  - i) Basis step: Verify P(b), P(b+1), ..., P(b+j) are true
  - ii)Inductive step: Show $P(b+1) \wedge P(b+2) \wedge \ldots \wedge P(b+k) \rightarrow P(k+1)$ is true for every integer $k \geq b+j$

- Example: Show that if n is an integer greater than 1, then n can be written as the product of primes
- Proof by strong induction:
  - First identify P(n), P(n): n can be written as the product of primes
  - Basis step: P(2): 2 is a prime number, so P(2) is true.
  - Inductive step:
    - Assume P(j) is true for $1 < j \leq k$ for an arbitrary $k > 1$, i.e. j can be written as the product of primes when $1 < j \leq k$
    - Need to show P(k+1).
      - Case 1: k+1 is prime, then P(k+1) is true.
      - Case 2: k+1 is composite

        $k+1 = a*b$, where a and b are positive integers, and $2 \leq a, b \leq k$. By the assumption, a and b can be written as the product of primes. Then k+1 can be written as the product of primes.