Math/CSE 1019C:
Discrete Mathematics for Computer Science
Fall 2012

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Course page:
http://www.cse.yorku.ca/course/1019

- No Assignment is released today!
- No Class on Thanks Giving! Oct 8th
- Test 1 on Oct 15th,
  - Ch1.1-1.8
  - 7pm-8:20pm
  - Location: SLH F
  - Lecture: 8:40pm, SLH A.

Review of Sets

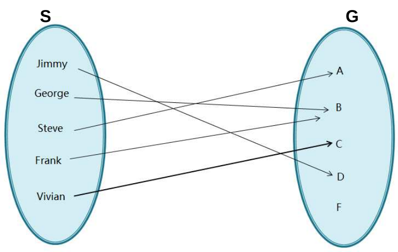
- What is a set?
  - Unordered collection of distinct elements
- How to describe a set?
  - Roster method: A={5,7,3}
  - set builder (predicates): S = {x | P(x)}
- Cardinality |S|
  - number of (distinct) elements |A| = 3

Exercises

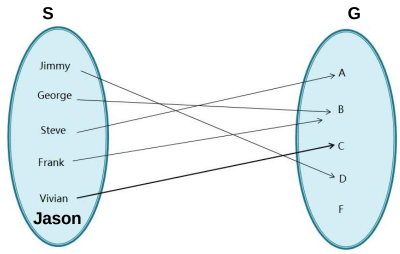
1. What is the cardinality of {∅,{∅,{∅}}}? What is its power set?
2. Prove that A ⊂ B iff P(A) ⊂P(B).
3. Draw the Venn Diagrams for $A \cap B \cap \bar{C} \cap D$

Review of Functions

- A function from A to B is an assignment of exactly one element of B to each element of A.
- grade: S → G

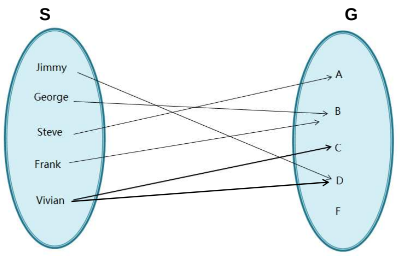


This is not a function!



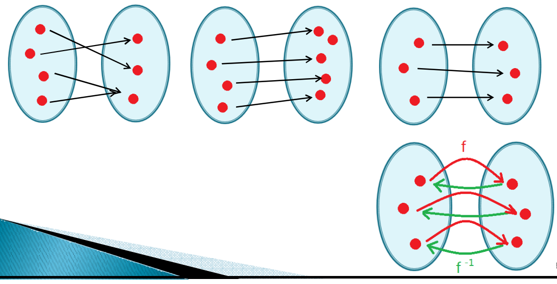
- Every member of the domain must be mapped to a member of the co-domain

This is not a function!



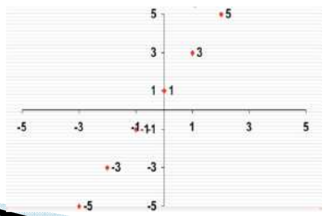
- No member of the domain may map to more than one member of the co-domain

- Surjections (onto)
- Injections (1-1)
- Bijections (1-1 correspondence): Invertible

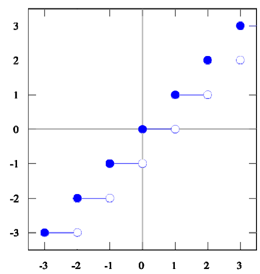


Graphs of Functions

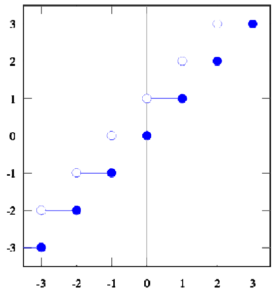
- Let f: A→B. The graph of f is the set of ordered pairs {(a,b) | a∈A and f(a)=b}
- Example: The graph of f:Z->Z where f(x)=2x+1



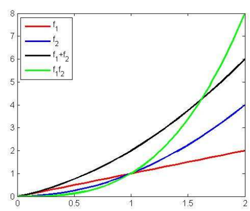
- The Graph of Floor function R->Z
  - $\lfloor x \rfloor$ is the largest integer that is less than or equal to x.



- The Graph of Ceiling function R->Z
  - $\lceil x \rceil$ is the smallest integer that is greater than or equal to x.



- Let f and g be functions from A to R. Then f_1+f_2 and f_1f_2 are also functions from A to R
  - $(f_1+f_2)(x) = f_1(x) + f_2(x)$
  - $(f_1f_2)(x) = f_1(x)\, f_2(x)$
- Example:
  - $f_1(x)=x$, $f_2(x)=x^2$
  - $(f_1+f_2)(x)=x + x^2$
  - $(f_1f_2)(x)= x^3$



- Notice the difference between $f \circ g$ and fg

Monotonic Functions

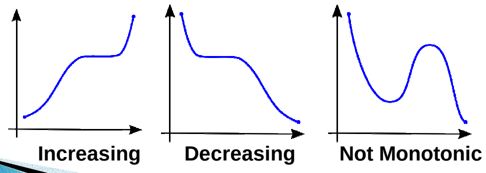
The domain and codomain of f are subsets of R.
x, y are in the domain of f and x<y.

f is (monotonically) increasing if $f(x) \le f(y)$

f is strictly increasing if $f(x) < f(y)$

f is (monotonically) decreasing if $f(x) \ge f(y)$

f is strictly decreasing if $f(x) > f(y)$



More Exercises for functions

- Show that $\lceil x+n \rceil$ is $\lceil x \rceil + n$ for $x \in R$ and $n \in Z$.
- Proof:
  - Assume $\lceil x \rceil = m$.
  - $m-1 < x \le m$
  - $n+m-1 < x+n \le m+n$
  - $\lceil x+n \rceil = m+n = \lceil x \rceil + n$
  - Q.E.D.

More Exercises for functions

- Show that $\lfloor 2x \rfloor$ is $\lfloor x \rfloor + \lfloor x+1/2 \rfloor$ for $x \in R$.
- Proof:
  - Assume $x=n+e$ where $n \in Z$, $e \in R$ and $0 \le e < 1$.
  - Case 1: $0 \le e < 1/2$
  - $\lfloor 2x \rfloor = \lfloor 2n+2e \rfloor = 2n$ $(0 \le 2e < 1)$
  - $\lfloor x \rfloor = \lfloor n+e \rfloor = n$ $(0 \le e < 1/2)$
  - $\lfloor x+1/2 \rfloor = \lfloor n+e+1/2 \rfloor = n$ $(1/2 \le e+1/2 < 1)$
  - So, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+1/2 \rfloor$
  - Case 2: $1/2 \le e < 1$
  - $\lfloor 2x \rfloor = \lfloor 2n+2e \rfloor = 2n+1$ $(1 \le 2e < 2)$
  - $\lfloor x \rfloor = \lfloor n+e \rfloor = n$ $(1/2 \le e < 1)$
  - $\lfloor x+1/2 \rfloor = \lfloor n+e+1/2 \rfloor = n+1$ $(1 \le e+1/2 < 1\ 1/2)$

More Exercises for functions

- Changing bases: In general need to go through the decimal representation
- E.g: $101_7 = ?_9$
- $101_7 = 1*7^2 + 0*7^1 + 1*7^0 = 50$
- Decimal to Base 9:
- d_1 = n rem 9 = 5, n = n div 9 = 5
- b_2 = n rem 9 = 5, n = n div 9 = 0.
- STOP
- So $101_7 = 55_9$

More Exercises for functions

- Changing bases that are powers of 2:
- Can often use shortcuts.
- Binary to Octal:
- 10|111|101 $= 275_8$

- Binary to Hexadecimal:
- 1011|1101 $= BD_{16}$

- Hexadecimal to Octal: Go through binary, not decimal.

More Exercises for functions

- 1. Prove that a strictly increasing function from R to itself is one to one
- 2. Suppose that f:Y->Z and g:X->Y are invertible. Show that

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

Sequences

- A sequence is an ordered list, possibly infinite, of elements

  notated by $\{a_1, a_2, a_3 \ldots\}$ or $\{a_i\}_{i=1}^{k}$

  where k is the upper limit (usually ∞)
- A sequence is a function from a subset of the Z (usually $\{0,1,2,\ldots\}$) to another set
- a_n is the image of the the integer n. We call a_n a term of the sequence, and n is its index or subscript

Arithmetic Progressions

- An arithmetic progression is a sequence of the form

  $a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots$

  a is the initial term

  d is the common difference
- E.g.
  - $\{-1, 3, 7, 11, \ldots\}$
  - $\{7,4,1,-2, \ldots\}$

Geometric Progressions

- An geometric progression is a sequence of the form

  $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$

  a is the initial term

  r is the common ratio
- E.g.
  - $\{1,-1,1,-1,1, \ldots\}$
  - $\{2,10,50,250,1250, \ldots\}$

Useful Sequences

- $\{n^2\}$: 1, 4, 9, 16, 25, ...
- $\{n^3\}$: 1, 8, 27, 64, 125, ...
- $\{n^4\}$: 1, 16, 81, 256, 343, ...
- $\{2^n\}$: 2, 4, 8, 16, 32, ...
- $\{3^n\}$: 3, 9, 27, 81, 243, ...
- $\{n!\}$: 1, 2, 6, 24, 120, ...
- $\{f_n\}$: 1, 1, 2, 3, 5, 8, …

Series

- A series is the sum of the terms of a sequence

  $S = a_1 + a_2 + a_3 + a_4 + \ldots$
- Consider the sequence $S_1, S_2, S_3, \ldots S_n$, where $S_i = a_1 + a_2 + \ldots + a_i$

In general we would like to evaluate sums of series – useful in algorithm analysis.

e.g. what is the total time spent in a nested loop?

Summations

- Given a sequence $\{a_i\}$ the summation notation for its terms $a_m, a_{m+1}, \ldots, a_n$

  $$\sum_{i=m}^{n} a_i \ , \ \sum_{i=m}^{n} a_i \ , \ \text{or} \ \sum_{m \le i \le n} a_i$$

  represent $a_m + a_{m+1} + \ldots + a_n$
- E.g.

  $$r^0 + r^1 + r^2 + \ldots + r^n = \sum_{i=1}^{n} r^i$$

  $$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + = \sum_{i=1}^{\infty} \frac{i}{i+1}$$

Summation of Arithmetic Progression

- Given a arithmetic progression a, a+d, a+2d, a+3d, . . ., a+nd, its summation is

$$\sum_{i=0}^{n}(a+id)=\frac{(2a+nd)(n+1)}{2}$$

- Proof on board
- You should also be able to determine the sum if the index starts at k and/or ends at n-1, n+1, etc.
- Page 166: useful summation formula

Summation of Geometric Progression

- Given a geometric progression a, ar^2, ar^3, ... , ar^n, its summation is

$$\sum_{i=0}^{n} ar^i = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Proof on board
- You should also be able to determine the sum if the index starts at k and/or ends at n-1, n+1, etc.

Infinite series

- Let x be a real number with |x|<1. Find

$$\sum_{i=0}^{\infty} x^i$$

- How about |x|≥1?
- Need to be very careful with infinite series
- In general, tools from calculus are needed to know whether an infinite series sum exists.

Double Summation

$$\sum_{i=1}^{4}\sum_{j=1}^{3}(ij)$$
$$=\sum_{i=1}^{4}(i+2i+3i)$$
$$=\sum_{i=1}^{4}(6i)$$
$$=6+12+18+24=60$$

- loop 1: for i=1 to 4
- loop 2: for j=1 to 3
- S = S + ij

Cardinality

- Recall: A set is finite if its cardinality is some (finite) integer n
- For two sets A and B
  - |A| = |B| if and only if there is a bijection from A to B
  - |A| ≤ |B| if there is an injection from A to B
  - |A| = |B| if |A| ≤ |B| and |B| ≤ |A|
  - |A| ≤ |B| if A⊆B

Infinite sets

- Why do we care?
- Cardinality of infinite sets
- Do all infinite sets have the same cardinality?

Countability

- A set is countable if
  - it is finite or
  - it has the same cardinality as the set of the positive integers Z^+ i.e. $|A| = |Z^+|$. The set is countably infinite
- We write $|A| = |Z+| = \aleph_0 =$ aleph null
- A set that is not countable is called uncountable

- Countability implies that there is a listing of the elements of the set.
- Fact (Will not prove): Any subset of a countable set is countable.
- Proving the set is countable involves (usually) constructing an explicit bijection with Z^+

- Show that the set of odd positive integers S is countable.
- Proof:
  - To show that S is countable, we will show a bijective function between Z^+ and S.
  - Consider f: Z^+ ->S be such that f(n) = 2n-1.
  - To see f is one-to-one, suppose that f(n)=f(m), then 2n-1=2m-1, so n=m.
  - To see f is onto, suppose that $t \in S$, i.e. t=2k-1 for some positive integer k. Hence t=f(k).
  - Q.E.D.

The integers are countable

- Write them as
  0, 1, −1, 2, −2, 3, −3, 4, −4, ……
- Find a bijection between this sequence and 1,2,3,4,…..
  Notice the pattern:
  $1 \to 0$ $2 \to 1$
  $3 \to -1$ $4 \to 2$
  $5 \to -2$ $6 \to 3$

So f(n) = n/2 if n even
−(n−1)/2 o.w.

Other simple bijections

- Union of two countable sets A, B is countable:
  Say f: $N \to A$, g:$N \to B$ are bijections
  New bijection h: $N \to A \cup B$
  h(n) = f(n/2) if n is even
  = g((n−1)/2) if n is odd.

The rationals are countable

- Step 1. Show that $Z^+ \times Z^+$ is countable.
- Step 2. Show injection between Q^+, $Z^+ \times Z^+$.
- Step 3. Construct a bijection from Q^+ to Q